SEIK - the unknown ensemble Kalman filter

Lars Nerger, Tijana Janjić, Wolfgang Hiller, and Jens Schröter

Alfred Wegener Institute for Polar and Marine Research, Bremerhaven, Germany

Contact: Lars.Nerger@awi.de · http://www.awi.de

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AWI Alfred Wegener Institute for Polar and Marine Research

Introduction

The SEIK filter (Singular "Evolutive" Interpolated Kalman filter) has been introduced in 1998 by Pham et al. [1] as a variant of the SEEK filter, which is a reduced-rank approximation of the Extended Kalman Filter. In recent years, it has been shown that the SEIK filter is an ensemble-based Kalman filter that uses a factorization rather than square-root of the state error covariance matrix, see e.g. [2]. Unfortunately, the existence of the SEIK filter as an ensemble-based Kalman filter with similar efficiency as the later introduced ensemble square-root Kalman filters, appears to be widely unknown and the SEIK filter is typically omitted in reviews about ensemble-based Kalman filters.

To raise the attention about the SEIK filter as a very efficient ensemble-based Kalman filter, we review the filter algorithm and compare it with ensemble square-root Kalman filter algorithms. For a practical comparison, the SEIK filter and the Ensemble Transformation Kalman filter (ETKF, [3]) are applied in twin experiments assimilating sea surface height data into the finite-element ocean model FEOM. The analytical comparison as well as the numerical experiments show that the SEIK filter is equivalent to the ETKF under certain conditions.

Comparison of Filters

The equations for the SEIK and ETKF algorithms are displayed on the right hand side. The equations are very similar, so care is necessary when comparing the algorithms.

- The main difference is that ETKF uses the ensemble perturbation matrix $\mathbf{Z}$ to represent the estimated error space while SEIK uses the basis of the error space in matrix $\mathbf{L}$, which has one column less than $\mathbf{Z}$.
- The transformation matrix $\mathbf{A}$ of the SEIK filter is smaller than $\tilde{\mathbf{A}}$ of ETKF by one row and one column. Nonetheless, both contain the same information on the error space.
- As the ensemble in the SEIK filter is reduced to the basis of the error space, the analysis ensemble has to be re-created from this information. This is performed by the matrix Ω.
- SEIK and ETKF compute the analysis state $\mathbf{x}^a$ using the same error space information. Due to this, the analysis states are identical, if the same forecast ensemble and the same set of observations is used.
- Also the analysis ensembles of both filter algorithms will be equal when a particular choice for the matrix Ω is used. This is obtained when the Householder reflection orthogonal to the vector $(1,\dots,1)^T$ is applied to the identity matrix.
- When Ω is chosen to be a random matrix, it serves for the randomization of the analysis ensemble which is sometimes suggested to avoid a loss of rank in the analysis ensemble.

Conclusion

- The SEIK filter is an ensemble square-root filter similar to the ETKF. While ETKF uses the ensemble perturbations to represent the error space, SEIK directly uses a basis of it.
- Under certain conditions SEIK and ETKF become equivalent in that they result in the same analysis state and ensemble. This is the case if both filters use the symmetric square root fo the transformation matrix ($\mathbf{A}, \tilde{\mathbf{A}}$ and SEIK uses a particular deterministic choice for its matrix Ω.
- An assimilation experiment in the North Atlantic showed no differences in the estimated state for both the SEIK and ETKF filters.

Filter Equations for SEIK and ETKF

(The equations mostly follow the notations of [4] and [5])

	SEIK	ETKF
Some definitions		
State vector	$\mathbf{x}^a \in \mathbb{R}^n$	equal to SEIK
Ensemble of N members	$\mathbf{X}^a = \left[\mathbf{x}^{a(1)},\dots,\mathbf{x}^{a(N)}\right], \quad \mathbf{X}^a \in \mathbb{R}^{n\times N}$	equal to SEIK
Perturbation matrix	$\mathbf{Z}^a = \mathbf{X}^a - \overline{\mathbf{X}^a}, \quad \overline{\mathbf{X}^a} = [\overline{\mathbf{x}}^a,\dots,\overline{\mathbf{x}}^a]$	equal to SEIK
Analysis covariance matrix	$\mathbf{P}^a = \frac{1}{N-1}\mathbf{Z}^a(\mathbf{Z}^a)^T$	equal to SEIK
Error subspace basis	$\mathbf{L}^f = \mathbf{X}^f\mathbf{T}, \quad \mathbf{L}^f \in \mathbb{R}^{n\times(N-1)}$	not used in ETKF
T-matrix	$\mathbf{T} = \begin{pmatrix}\mathbf{I}_{(N-1)\times(N-1)} \\ \mathbf{0}_{1\times(N-1)}\end{pmatrix} - \frac{1}{N}\left(\mathbf{1}_{N\times(N-1)}\right)$	not used in ETKF
Analysis covariance matrix	$\mathbf{P}^a = \mathbf{L}^f\mathbf{A}(\mathbf{L}^f)^T$	$\mathbf{P}^a = \mathbf{Z}^f\tilde{\mathbf{A}}(\mathbf{Z}^f)^T$
with transformation matrix	$\mathbf{A} \in \mathbb{R}^{(N-1)\times(N-1)}$	$\tilde{\mathbf{A}} \in \mathbb{R}^{N\times N}$
	$\mathbf{A}^{-1} = (N-1)\mathbf{T}^T\mathbf{T} + (\mathbf{H}\mathbf{L}^f)^T\mathbf{R}^{-1}\mathbf{H}\mathbf{L}^f$	$\tilde{\mathbf{A}}^{-1} = (N-1)\mathbf{I} + (\mathbf{H}\mathbf{Z}^f)^T\mathbf{R}^{-1}\mathbf{H}\mathbf{Z}^f$
State analysis		
	$\mathbf{x}^a = \overline{\mathbf{x}^f} + \mathbf{L}^f\overline{\mathbf{w}}^{SEIK}$	$\mathbf{x}^a = \overline{\mathbf{x}^f} + \mathbf{Z}^f\overline{\mathbf{w}}^{ETKF}$
with weight vector	$\overline{\mathbf{w}}^{SEIK} = \mathbf{A}(\mathbf{H}\mathbf{L}^f)^T\mathbf{R}^{-1}\left(\mathbf{y}^o - \mathbf{H}\overline{\mathbf{x}^f}\right)$	$\overline{\mathbf{w}}^{ETKF} = \tilde{\mathbf{A}}(\mathbf{H}\mathbf{Z}^f)^T\mathbf{R}^{-1}\left(\mathbf{y}^o - \mathbf{H}\overline{\mathbf{x}^f}\right)$
Square-root of analysis covariance matrix		
	$\mathbf{Z}^a = \mathbf{L}^f\mathbf{W}^{SEIK}$	$\mathbf{Z}^a = \mathbf{Z}^f\mathbf{W}^{ETKF}$
with weight matrix	$\mathbf{W}^{SEIK} = \sqrt{N-1}\mathbf{C}\Omega^T$	$\mathbf{W}^{ETKF} = \sqrt{N-1}\tilde{\mathbf{C}}$
and square-roots $\mathbf{C}$, $\tilde{\mathbf{C}}$	$\mathbf{C}\mathbf{C}^T = \mathbf{A}$	$\tilde{\mathbf{C}}\tilde{\mathbf{C}}^T = \tilde{\mathbf{A}}$
	$\mathbf{C}$ can be the symmetric square root $\mathbf{C} = \mathbf{U}\mathbf{S}^{-1/2}\mathbf{U}^T$ from the singular value decomposition $\mathbf{U}\mathbf{S}\mathbf{V} = \mathbf{A}^{-1}$. Alternatively, a Cholesky factorization can be used as square-root.	analogous to SEIK
Matrix Ω	Ω can be an arbitrary $N\times(N-1)$ matrix with orthogonal columns orthogonal to $(1,\dots,1)^T$.	
Ensemble transformation		
	$\mathbf{X}^a = \overline{\mathbf{X}^a} + \mathbf{L}^f\mathbf{W}^{SEIK}$	$\tilde{\mathbf{X}}^a = \overline{\mathbf{X}^a} + \mathbf{Z}^f\mathbf{W}^{ETKF}$
Localization		

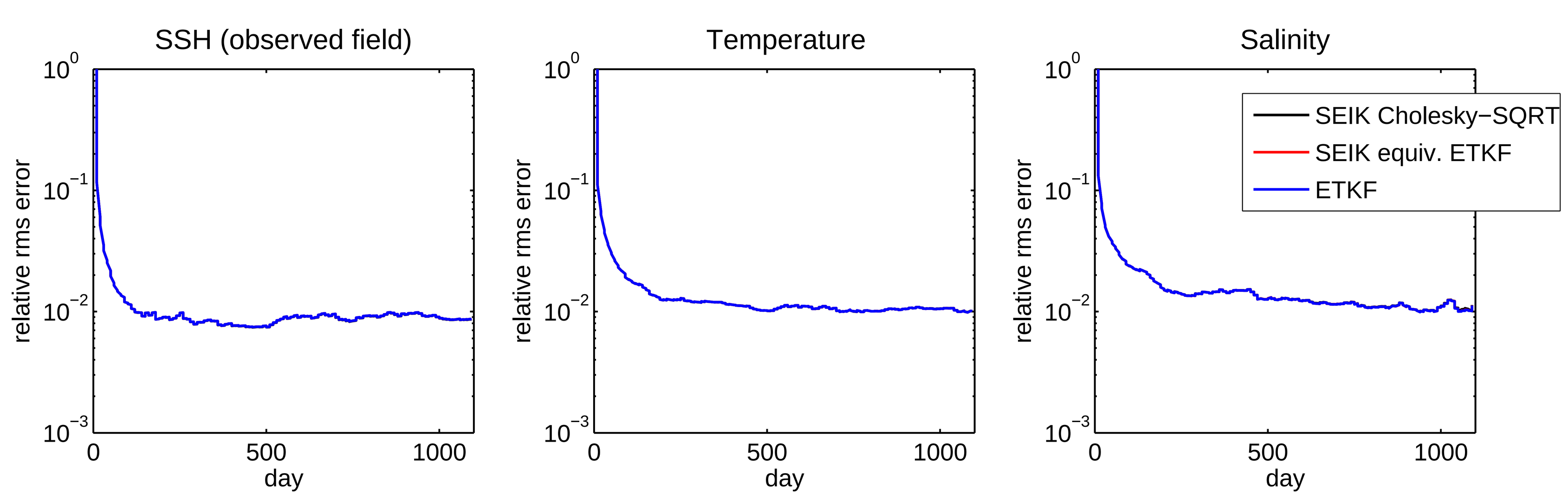
The localization can be performed in an identical way for SEIK and ETKF (see [6] and [7]) by applying a sequence of local updates with defined influence radius for the observations.

Assimilation Experiment

Twin experiments were conducted using the finite-element ocean model FEOM in a configuration for the North Atlantic. A triangular mesh with a horizontal resolution of 1° and 20 levels in the vertical is used.

The ETKF and the SEIK filter were used to assimilate synthetic observations of the sea surface height (SSH) each tenth day over three years. For SEIK, a configuration was used that makes it equivalent to ETKF (see box "Comparison of Filters") as well as a square-root based on Cholesky decompostion.

Ensemble sizes between 8 and 64 were tested, showing that more than 32 members did not further reduce the estimation errors. The global formulations of SEIK and ETKF were used. These were sufficient due to the coarse resolution of the model while localization required an almost global influence radius to be of comparable performance.



RMS errors relative to a free running ensemble forecast. The non-observed temperature and salinity fields are reduced by about the same amount as the observed sea surface height (SSH). The SEIK filter configured to be equivalent to ETKF provides an identical result to the ETKF. In addition, the result from the SEIK filter using a Cholesky decomposition of the transformation matrix $\mathbf{A}$ is identical to the result of the ETKF. This shows, that the potentially larger change in the ensemble members of the SEIK filter with Cholesky decomposition does not lead to an unstable forecast in this example.

References

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