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Definition of functionals of the geopotential used in GrafLab software

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Gravitational potential

$$V(r,\theta,\lambda) = \frac{GM}{r}\left[1 + \sum_{n=2}^{M}\left(\frac{R}{r}\right)^n \sum_{m=0}^{n}\left(\overline{C}_{n,m}\,\cos(m\lambda) + \overline{S}_{n,m}\,\sin(m\lambda)\right)\overline{P}_{n,m}(\cos\theta)\right] \tag{1}$$

Gravitational tensor in the spherical coordinates

$$\mathbf{V}(r,\theta,\lambda) = \begin{pmatrix} V_{rr} & V_{r\theta} & V_{r\lambda} \\ V_{\theta r} & V_{\theta\theta} & V_{\theta\lambda} \\ V_{\lambda r} & V_{\lambda\theta} & V_{\lambda\lambda} \end{pmatrix} \tag{2}$$

$$\begin{aligned} V_{rr}(r,\theta,\lambda) &= \frac{\partial^2 V(r,\theta,\lambda)}{\partial r^2} \\ &= \frac{GM}{r^3}\left[2 + \sum_{n=2}^{M}\left(\frac{R}{r}\right)^n (n+1)(n+2)\sum_{m=0}^{n}\left(\overline{C}_{n,m}\,\cos(m\lambda) + \overline{S}_{n,m}\,\sin(m\lambda)\right)\overline{P}_{n,m}(\cos\theta)\right] \end{aligned} \tag{3}$$

$$\begin{aligned} V_{r\theta}(r,\theta,\lambda) &= \frac{1}{r}\,\frac{\partial^2 V(r,\theta,\lambda)}{\partial r\,\partial\theta} \\ &= -\frac{GM}{r^3}\sum_{n=2}^{M}\left(\frac{R}{r}\right)^n (n+1)\sum_{m=0}^{n}\left(\overline{C}_{n,m}\,\cos(m\lambda) + \overline{S}_{n,m}\,\sin(m\lambda)\right)\frac{\mathrm{d}\overline{P}_{n,m}(\cos\theta)}{\mathrm{d}\theta} \end{aligned} \tag{4}$$

$$\begin{aligned} V_{r\lambda}(r,\theta,\lambda) &= \frac{1}{r\sin\theta}\,\frac{\partial^2 V(r,\theta,\lambda)}{\partial r\,\partial\lambda} \\ &= -\frac{GM}{r^3\sin\theta}\sum_{n=2}^{M}\left(\frac{R}{r}\right)^n (n+1)\sum_{m=0}^{n}\left(\overline{S}_{n,m}\,\cos(m\lambda) - \overline{C}_{n,m}\,\sin(m\lambda)\right) m\,\overline{P}_{n,m}(\cos\theta) \end{aligned} \tag{5}$$

$$\begin{aligned} V_{\theta\theta}(r,\theta,\lambda) &= \frac{1}{r^2}\,\frac{\partial^2 V(r,\theta,\lambda)}{\partial\theta^2} \\ &= \frac{GM}{r^3}\sum_{n=2}^{M}\left(\frac{R}{r}\right)^n \sum_{m=0}^{n}\left(\overline{C}_{n,m}\,\cos(m\lambda) + \overline{S}_{n,m}\,\sin(m\lambda)\right)\frac{\mathrm{d}^2\overline{P}_{n,m}(\cos\theta)}{\mathrm{d}\theta^2} \end{aligned} \tag{6}$$

$$\begin{aligned} V_{\theta\lambda}(r,\theta,\lambda) &= \frac{1}{r^2\sin\theta}\,\frac{\partial^2 V(r,\theta,\lambda)}{\partial\theta\,\partial\lambda} \\ &= \frac{GM}{r^3\sin\theta}\sum_{n=2}^{M}\left(\frac{R}{r}\right)^n \sum_{m=0}^{n}\left(\overline{S}_{n,m}\,\cos(m\lambda) - \overline{C}_{n,m}\,\sin(m\lambda)\right) m\,\frac{\mathrm{d}\overline{P}_{n,m}(\cos\theta)}{\mathrm{d}\theta} \end{aligned} \tag{7}$$

$$V_{\lambda\lambda}(r,\theta,\lambda) = \frac{1}{r^2 \sin^2\theta} \frac{\partial^2 V(r,\theta,\lambda)}{\partial \lambda^2}$$
$$= -\frac{GM}{r^3 \sin^2\theta} \sum_{n=2}^{M} \left(\frac{R}{r}\right)^n \sum_{m=0}^{n} \left(\overline{C}_{n,m} \cos(m\lambda) + \overline{S}_{n,m} \sin(m\lambda)\right) m^2 \, \overline{P}_{n,m}(\cos\theta) \tag{8}$$

Gravitational tensor in the local north-oriented reference frame[1]

$$\mathbf{V}(r,\theta,\lambda) = \begin{pmatrix} V_{xx} & V_{xy} & V_{xz} \\ V_{yx} & V_{yy} & V_{yz} \\ V_{zx} & V_{zy} & V_{zz} \end{pmatrix} \tag{9}$$

$$V_{xx} = -\frac{GM}{r^3} + \frac{GM}{r^3} \sum_{n=2}^{M} \left(\frac{R}{r}\right)^n \sum_{m=-n}^{n} \overline{C}_{n,m} \, Q_m(\lambda) \Big(a_{n,m} \, \overline{P}_{n,|m|-2}(\cos\theta)$$
$$+ \left[b_{n,m} - (n+1)(n+2)\right] \overline{P}_{n,|m|}(\cos\theta) + c_{n,m} \, \overline{P}_{n,|m|+2}(\cos\theta) \Big) \tag{10}$$

$$V_{yy} = -\frac{GM}{r^3} - \frac{GM}{r^3} \sum_{n=2}^{M} \left(\frac{R}{r}\right)^n \sum_{m=-n}^{n} \overline{C}_{n,m} \, Q_m(\lambda) \Big(a_{n,m} \, \overline{P}_{n,|m|-2}(\cos\theta)$$
$$+ b_{n,m} \, \overline{P}_{n,|m|}(\cos\theta) + c_{n,m} \, \overline{P}_{n,|m|+2}(\cos\theta) \Big) \tag{11}$$

$$V_{xy} = \frac{GM}{r^3} \sum_{n=2}^{M} \left(\frac{R}{r}\right)^n \sum_{m=-n}^{n} \overline{C}_{n,m} \, Q_{-m}(\lambda) \Big(d_{n,m} \, \overline{P}_{n-1,|m|-2}(\cos\theta)$$
$$+ g_{n,m} \, \overline{P}_{n-1,|m|}(\cos\theta) + h_{n,m} \, \overline{P}_{n-1,|m|+2}(\cos\theta) \Big), \quad m \neq 0 \tag{12}$$

$$V_{xz} = \frac{GM}{r^3} \sum_{n=2}^{M} \left(\frac{R}{r}\right)^n \sum_{m=-n}^{n} \overline{C}_{n,m} \, Q_m(\lambda) \Big(\beta_{n,m} \, \overline{P}_{n,|m|-1}(\cos\theta)$$
$$+ \gamma_{n,m} \, \overline{P}_{n,|m|+1}(\cos\theta) \Big) \tag{13}$$

$$V_{yz} = \frac{GM}{r^3} \sum_{n=2}^{M} \left(\frac{R}{r}\right)^n \sum_{m=-n}^{n} \overline{C}_{n,m} \, Q_{-m}(\lambda) \Big(\mu_{n,m} \, \overline{P}_{n-1,|m|-1}(\cos\theta)$$
$$+ \nu_{n,m} \, \overline{P}_{n-1,|m|+1}(\cos\theta) \Big), \quad m \neq 0 \tag{14}$$

$$V_{zz} = 2\frac{GM}{r^3} + \frac{GM}{r^3} \sum_{n=2}^{M} \left(\frac{R}{r}\right)^n (n+1)(n+2) \sum_{m=-n}^{n} \overline{C}_{n,m} \, Q_m(\lambda) \, \overline{P}_{n,|m|}(\cos\theta) \tag{15}$$

where

$$Q_m(\lambda) = \begin{cases} \cos m\lambda, & m \geq 0 \\ \sin |m|\lambda, & m < 0 \end{cases} \tag{16}$$

[1] In the GrafLab, Eq. (10) - Eq. (15) have been slightly modified, see appendix A.

$$a_{n,m} = 0, \quad |m| = 0,1 \tag{17}$$

$$a_{n,m} = \frac{\sqrt{1+\delta_{|m|,2}}}{4}\sqrt{n^2-(|m|-1)^2}\sqrt{n+|m|}\sqrt{n-|m|+2}, \quad 2 \le |m| \le n \tag{18}$$

$$b_{n,m} = \frac{(n+|m|+1)(n+|m|+2)}{2\,(|m|+1)}, \quad |m| = 0,1 \tag{19}$$

$$b_{n,m} = \frac{n^2+m^2+3n+2}{2}, \quad 2 \le |m| \le n \tag{20}$$

$$c_{n,m} = \frac{\sqrt{1+\delta_{|m|,0}}}{4}\sqrt{n^2-(|m|+1)^2}\sqrt{n-|m|}\sqrt{n+|m|+2}, \quad |m| = 0,1 \tag{21}$$

$$c_{n,m} = \frac{1}{4}\sqrt{n^2-(|m|+1)^2}\sqrt{n-|m|}\sqrt{n+|m|+2}, \quad 2 \le |m| \le n \tag{22}$$

$$d_{n,m} = 0, \quad |m| = 1 \tag{23}$$

$$\begin{aligned} d_{n,m} = &-\frac{m}{4|m|}\sqrt{\frac{2n+1}{2n-1}}\sqrt{1+\delta_{|m|,2}}\sqrt{n^2-(|m|-1)^2} \\ &\times \sqrt{n+|m|}\sqrt{n+|m|-2}, \quad 2 \le |m| \le n \end{aligned} \tag{24}$$

$$g_{n,m} = \frac{m}{4|m|}\sqrt{\frac{2n+1}{2n-1}}\sqrt{n+1}\sqrt{n-1}(n+2), \quad |m| = 1 \tag{25}$$

$$g_{n,m} = \frac{m}{2}\sqrt{\frac{2n+1}{2n-1}}\sqrt{n+|m|}\sqrt{n-|m|}, \quad 2 \le |m| \le n \tag{26}$$

$$h_{n,m} = \frac{m}{4|m|}\sqrt{\frac{2n+1}{2n-1}}\sqrt{n-3}\sqrt{n-2}\sqrt{n-1}\sqrt{n+2}, \quad |m| = 1 \tag{27}$$

$$h_{n,m} = \frac{m}{4|m|}\sqrt{\frac{2n+1}{2n-1}}\sqrt{n^2-(|m|+1)^2}\sqrt{n-|m|}\sqrt{n-|m|-2}, \quad 2 \le |m| \le n \tag{28}$$

$$\beta_{n,m} = 0, \quad m = 0 \tag{29}$$

$$\beta_{n,m} = \frac{n+2}{2}\sqrt{1+\delta_{|m|,1}}\sqrt{n+|m|}\sqrt{n-|m|+1}, \quad 1 \le |m| \le n \tag{30}$$

$$\gamma_{n,m} = -(n+2)\sqrt{\frac{n(n+1)}{2}}, \quad m = 0 \tag{31}$$

$$\gamma_{n,m} = -\frac{n+2}{2}\sqrt{n-|m|}\sqrt{n+|m|+1}, \quad 1 \le |m| \le n \tag{32}$$

$$\mu_{n,m} = -\frac{m}{|m|}\left(\frac{n+2}{2}\right)\sqrt{\frac{2n+1}{2n-1}}\sqrt{1+\delta_{|m|,1}}\sqrt{n+|m|}\sqrt{n+|m|-1} \tag{33}$$

$$\nu_{n,m} = -\frac{m}{|m|}\left(\frac{n+2}{2}\right)\sqrt{\frac{2n+1}{2n-1}}\sqrt{n-|m|}\sqrt{n-|m|-1} \tag{34}$$

$$\delta_{p,q} = \begin{cases} 1, & p = q, \\ 0, & p \ne q. \end{cases} \tag{35}$$

Gravity potential

$$W(r,\theta,\lambda) = \frac{GM}{r}\left[1 + \sum_{n=2}^{M}\left(\frac{R}{r}\right)^{n}\sum_{m=0}^{n}\left(\overline{C}_{n,m}\,\cos(m\lambda) + \overline{S}_{n,m}\,\sin(m\lambda)\right)\overline{P}_{n,m}(\cos\theta)\right] + \frac{1}{2}\,\omega^2\,r^2\,\sin^2\theta \tag{36}$$

Gravity

$$\begin{aligned} g(r,\theta,\lambda) &= |\nabla W(r,\theta,\lambda)| \\ &= \sqrt{\left(\frac{\partial V}{\partial r} + \frac{\partial V_c}{\partial r}\right)^2 + \left[\frac{1}{r}\left(\frac{\partial V}{\partial \theta} + \frac{\partial V_c}{\partial \theta}\right)\right]^2 + \left[\frac{1}{r\sin\theta}\left(\frac{\partial V}{\partial \lambda} + \frac{\partial V_c}{\partial \lambda}\right)\right]^2}, \end{aligned} \tag{37}$$

$$\begin{aligned} \frac{\partial V}{\partial r} &= -\frac{GM}{r^2}\left[1 + \sum_{n=2}^{M}\left(\frac{R}{r}\right)^{n}(n+1)\sum_{m=0}^{n}\left(\overline{C}_{n,m}\,\cos(m\lambda) + \overline{S}_{n,m}\,\sin(m\lambda)\right)\overline{P}_{n,m}(\cos\theta)\right] \\ \frac{\partial V}{\partial \theta} &= \frac{GM}{r}\sum_{n=2}^{M}\left(\frac{R}{r}\right)^{n}\sum_{m=0}^{n}\left(\overline{C}_{n,m}\,\cos(m\lambda) + \overline{S}_{n,m}\,\sin(m\lambda)\right)\frac{\mathrm{d}\overline{P}_{n,m}(\cos\theta)}{\mathrm{d}\theta} \\ \frac{\partial V}{\partial \lambda} &= \frac{GM}{r}\sum_{n=2}^{M}\left(\frac{R}{r}\right)^{n}\sum_{m=0}^{n}\left(\overline{S}_{n,m}\,\cos(m\lambda) - \overline{C}_{n,m}\,\sin(m\lambda)\right)m\,\overline{P}_{n,m}(\cos\theta) \end{aligned} \tag{38}$$

$$V_c = \frac{1}{2}\,\omega^2\,r^2\,\sin^2\theta \tag{39}$$

$$\frac{\partial V_c}{\partial r} = \omega^2\,r\,\sin^2\theta, \quad \frac{\partial V_c}{\partial \theta} = \omega^2\,r^2\,\sin\theta\,\cos\theta, \quad \frac{\partial V_c}{\partial \lambda} = 0 \tag{40}$$

Gravity sa (spherical approximation)

$$g_{sa}(r,\theta,\lambda) = \sqrt{\left(\frac{\partial V}{\partial r} + \frac{\partial V_c}{\partial r}\right)^2} \tag{41}$$

Second radial derivative of gravity potential

$$\frac{\partial^2 W(r,\theta,\lambda)}{\partial r^2} = \frac{\partial^2 V(r,\theta,\lambda)}{\partial r^2} + \frac{\partial^2 V_c(r,\theta,\lambda)}{\partial r^2} \tag{42}$$

$$\frac{\partial^2 V(r,\theta,\lambda)}{\partial r^2}$$

$$= \frac{GM}{r^3}\left[2 + \sum_{n=2}^{M}\left(\frac{R}{r}\right)^n (n+1)(n+2)\sum_{m=0}^{n}\left(\overline{C}_{n,m}\cos(m\lambda) + \overline{S}_{n,m}\sin(m\lambda)\right)\overline{P}_{n,m}(\cos\theta)\right] \tag{43}$$

$$\frac{\partial^2 V_c(r,\theta,\lambda)}{\partial r^2} = \omega^2 \sin^2\theta \tag{44}$$

Disturbing potential

$$T(r,\theta,\lambda) = \frac{GM}{r}\sum_{n=2}^{M}\left(\frac{R}{r}\right)^n \sum_{m=0}^{n}\left(\Delta\overline{C}_{n,m}\cos(m\lambda) + \Delta\overline{S}_{n,m}\sin(m\lambda)\right)\overline{P}_{n,m}(\cos\theta) \tag{45}$$

$$\Delta\overline{C}_{n,m} = \overline{C}_{n,m} - \overline{C}^{Ell}_{n,m}\frac{GM^{Ell}}{GM}\left(\frac{a^{Ell}}{R}\right)^n \tag{46}$$

$$\Delta\overline{S}_{n,m} = \overline{S}_{n,m} - \overline{S}^{Ell}_{n,m} = \overline{S}_{n,m} \tag{47}$$

Gravity disturbance

$$\delta g(r,\theta,\lambda) = g(r,\theta,\lambda) - \gamma_{SH}(r,\theta) \tag{48}$$

in which

- $\gamma_{SH}(r,\theta,\lambda)$ is the normal gravity evaluated from the spherical harmonics. The same formulae as for $g(r,\theta,\lambda)$ hold for computing $\gamma_{SH}(r,\theta,\lambda)$ by replacing $\overline{C}_{n,m}$, $\overline{S}_{n,m}$ by $\overline{C}^{Ell}_{n,m}$, $\overline{S}^{Ell}_{n,m}$ in eq. (37) – (38).

Gravity disturbance sa (spherical approximation)

$$\delta g_{sa}(r,\theta,\lambda) = -\frac{\partial T(r,\theta,\lambda)}{\partial r}$$

$$= \frac{GM}{r^2}\sum_{n=2}^{M}\left(\frac{R}{r}\right)^n (n+1)\sum_{m=0}^{n}\left(\Delta\overline{C}_{n,m}\cos(m\lambda) + \Delta\overline{S}_{n,m}\sin(m\lambda)\right)\overline{P}_{n,m}(\cos\theta) \tag{49}$$

Gravity anomaly sa (spherical approximation)

$$\Delta g_{sa}(r,\theta,\lambda) = -\frac{\partial T(r,\theta,\lambda)}{\partial r} - \frac{2}{r}T(r,\theta,\lambda)$$

$$= \frac{GM}{r^2}\sum_{n=2}^{M}\left(\frac{R}{r}\right)^n (n-1)\sum_{m=0}^{n}\big(\Delta\overline{C}_{n,m}\,\cos(m\lambda) + \Delta\overline{S}_{n,m}\,\sin(m\lambda)\big)\,\overline{P}_{n,m}(\cos\theta) \tag{50}$$

Second radial derivative of disturbing potential

$$T_{rr}(r,\theta,\lambda) = \frac{\partial^2 T(r,\theta,\lambda)}{\partial r^2}$$

$$= \frac{GM}{r^3}\sum_{n=2}^{M}\left(\frac{R}{r}\right)^n (n+1)(n+2)\sum_{m=0}^{n}\big(\Delta\overline{C}_{n,m}\,\cos(m\lambda) + \Delta\overline{S}_{n,m}\,\sin(m\lambda)\big)\,\overline{P}_{n,m}(\cos\theta) \tag{51}$$

Disturbing tensor in the spherical coordinates

$$\mathbf{T}(r,\theta,\lambda) = \begin{pmatrix} T_{rr} & T_{r\theta} & T_{r\lambda} \\ T_{\theta r} & T_{\theta\theta} & T_{\theta\lambda} \\ T_{\lambda r} & T_{\lambda\theta} & T_{\lambda\lambda} \end{pmatrix} \tag{52}$$

$$T_{rr}(r,\theta,\lambda) = \frac{\partial^2 T(r,\theta,\lambda)}{\partial r^2}$$

$$= \frac{GM}{r^3}\sum_{n=2}^{M}\left(\frac{R}{r}\right)^n (n+1)(n+2)\sum_{m=0}^{n}\big(\Delta\overline{C}_{n,m}\,\cos(m\lambda) + \Delta\overline{S}_{n,m}\,\sin(m\lambda)\big)\,\overline{P}_{n,m}(\cos\theta) \tag{53}$$

$$T_{r\theta}(r,\theta,\lambda) = \frac{1}{r}\,\frac{\partial^2 T(r,\theta,\lambda)}{\partial r\,\partial\theta}$$

$$= -\frac{GM}{r^3}\sum_{n=2}^{M}\left(\frac{R}{r}\right)^n (n+1)\sum_{m=0}^{n}\big(\Delta\overline{C}_{n,m}\,\cos(m\lambda) + \Delta\overline{S}_{n,m}\,\sin(m\lambda)\big)\,\frac{\mathrm{d}\overline{P}_{n,m}(\cos\theta)}{\mathrm{d}\theta} \tag{54}$$

$$T_{r\lambda}(r,\theta,\lambda) = \frac{1}{r\sin\theta}\,\frac{\partial^2 T(r,\theta,\lambda)}{\partial r\,\partial\lambda}$$

$$= -\frac{GM}{r^3\,\sin\theta}\sum_{n=2}^{M}\left(\frac{R}{r}\right)^n (n+1)\sum_{m=0}^{n}\big(\Delta\overline{S}_{n,m}\,\cos(m\lambda) - \Delta\overline{C}_{n,m}\,\sin(m\lambda)\big)m\,\overline{P}_{n,m}(\cos\theta) \tag{55}$$

$$\begin{aligned} T_{\theta\theta}(r,\theta,\lambda) &= \frac{1}{r^2}\,\frac{\partial^2 T(r,\theta,\lambda)}{\partial\theta^2} \\ &= \frac{GM}{r^3}\sum_{n=2}^{M}\left(\frac{R}{r}\right)^n\sum_{m=0}^{n}\big(\Delta\overline{C}_{n,m}\,\cos(m\lambda) + \Delta\overline{S}_{n,m}\,\sin(m\lambda)\big)\,\frac{\mathrm{d}^2\overline{P}_{n,m}(\cos\theta)}{\mathrm{d}\theta^2} \end{aligned} \tag{56}$$

$$T_{\theta\lambda}(r,\theta,\lambda) = \frac{1}{r^2\sin\theta}\,\frac{\partial^2 T(r,\theta,\lambda)}{\partial\theta\,\partial\lambda}$$

$$= \frac{GM}{r^3\,\sin\theta}\sum_{n=2}^{M}\left(\frac{R}{r}\right)^n\sum_{m=0}^{n}\big(\Delta\overline{S}_{n,m}\,\cos(m\lambda) - \Delta\overline{C}_{n,m}\,\sin(m\lambda)\big)\,m\,\frac{\mathrm{d}\overline{P}_{n,m}(\cos\theta)}{\mathrm{d}\theta} \tag{57}$$

$$T_{\lambda\lambda}(r,\theta,\lambda) = \frac{1}{r^2\sin^2\theta}\,\frac{\partial^2 T(r,\theta,\lambda)}{\partial\lambda^2}$$

$$= -\frac{GM}{r^3\,\sin^2\theta}\sum_{n=2}^{M}\left(\frac{R}{r}\right)^n\sum_{m=0}^{n}\big(\Delta\overline{C}_{n,m}\,\cos(m\lambda) + \Delta\overline{S}_{n,m}\,\sin(m\lambda)\big)\,m^2\,\overline{P}_{n,m}(\cos\theta) \tag{58}$$

Disturbing tensor in the local north-oriented reference frame[2]

$$\mathbf{T}(r,\theta,\lambda) = \begin{pmatrix} T_{xx} & T_{xy} & T_{xz} \\ T_{yx} & T_{yy} & T_{yz} \\ T_{zx} & T_{zy} & T_{zz} \end{pmatrix} \tag{59}$$

$$\begin{aligned} T_{xx} =& \frac{GM}{r^3}\sum_{n=2}^{M}\left(\frac{R}{r}\right)^n\sum_{m=-n}^{n}\overline{C}_{n,m}\,Q_m(\lambda)\Big(a_{n,m}\,\overline{P}_{n,|m|-2}(\cos\theta) \\ &+ [b_{n,m} - (n+1)(n+2)]\,\overline{P}_{n,|m|}(\cos\theta) + c_{n,m}\,\overline{P}_{n,|m|+2}(\cos\theta)\Big) \end{aligned} \tag{60}$$

[2] In the GrafLab, Eq. (60) - Eq. (65) have been slightly modified, see appendix A.

$$
\begin{aligned}
T_{yy} = &-\frac{GM}{r^3}\sum_{n=2}^{M}\left(\frac{R}{r}\right)^n \sum_{m=-n}^{n} \overline{C}_{n,m}\, Q_m(\lambda)\Big(a_{n,m}\,\overline{P}_{n,|m|-2}(\cos\theta) \\
&+ b_{n,m}\,\overline{P}_{n,|m|}(\cos\theta) + c_{n,m}\,\overline{P}_{n,|m|+2}(\cos\theta)\Big)
\end{aligned} \tag{61}
$$

$$
\begin{aligned}
T_{xy} = &\frac{GM}{r^3}\sum_{n=2}^{M}\left(\frac{R}{r}\right)^n \sum_{m=-n}^{n} \overline{C}_{n,m}\, Q_{-m}(\lambda)\Big(d_{n,m}\,\overline{P}_{n-1,|m|-2}(\cos\theta) \\
&+ g_{n,m}\,\overline{P}_{n-1,|m|}(\cos\theta) + h_{n,m}\,\overline{P}_{n-1,|m|+2}(\cos\theta)\Big), \quad m \neq 0
\end{aligned} \tag{62}
$$

$$
\begin{aligned}
T_{xz} = &\frac{GM}{r^3}\sum_{n=2}^{M}\left(\frac{R}{r}\right)^n \sum_{m=-n}^{n} \overline{C}_{n,m}\, Q_m(\lambda)\Big(\beta_{n,m}\,\overline{P}_{n,|m|-1}(\cos\theta) \\
&+ \gamma_{n,m}\,\overline{P}_{n,|m|+1}(\cos\theta)\Big)
\end{aligned} \tag{63}
$$

$$
\begin{aligned}
T_{yz} = &\frac{GM}{r^3}\sum_{n=2}^{M}\left(\frac{R}{r}\right)^n \sum_{m=-n}^{n} \overline{C}_{n,m}\, Q_{-m}(\lambda)\Big(\mu_{n,m}\,\overline{P}_{n-1,|m|-1}(\cos\theta) \\
&+ \nu_{n,m}\,\overline{P}_{n-1,|m|+1}(\cos\theta)\Big), \quad m \neq 0
\end{aligned} \tag{64}
$$

$$
T_{zz} = \frac{GM}{r^3}\sum_{n=2}^{M}\left(\frac{R}{r}\right)^n (n+1)(n+2) \sum_{m=-n}^{n} \overline{C}_{n,m}\, Q_m(\lambda)\,\overline{P}_{n,|m|}(\cos\theta) \tag{65}
$$

Deflections of the vertical

$$
\begin{aligned}
\xi(r,\theta,\lambda) = &-\frac{GM}{r^2\,\gamma(r,\theta)}\sum_{n=2}^{M}\left(\frac{R}{r}\right)^n \sum_{m=0}^{n}\big(\Delta\overline{C}_{n,m}\,\cos(m\lambda) \\
&+ \Delta\overline{S}_{n,m}\,\sin(m\lambda)\big)\frac{\mathrm{d}\overline{P}_{n,m}(\cos\theta)}{\mathrm{d}\theta}
\end{aligned} \tag{66}
$$

$$
\begin{aligned}
\eta(r,\theta,\lambda) = &-\frac{GM}{r^2\,\gamma(r,\theta)\sin\theta}\sum_{n=2}^{M}\left(\frac{R}{r}\right)^n \sum_{m=0}^{n}\big(\Delta\overline{S}_{n,m}\,\cos(m\lambda) \\
&- \Delta\overline{C}_{n,m}\,\sin(m\lambda)\big)m\,\overline{P}_{n,m}(\cos\theta)
\end{aligned} \tag{67}
$$

$$
\Theta(r,\theta,\lambda) = \sqrt{\xi^2(r,\theta,\lambda) + \eta^2(r,\theta,\lambda)} \tag{68}
$$

Geoid undulation

$$
H(\theta,\lambda) = \sum_{n=0}^{M}\sum_{m=0}^{n}\big(\overline{HC}_{n,m}\,\cos(m\lambda) + \overline{HS}_{n,m}\,\sin(m\lambda)\big)\,\overline{P}_{n,m}(\cos\theta) \tag{69}
$$

$$N(\theta,\lambda) = \frac{T(r_{ell},\theta,\lambda) - 2\pi\, G\, \rho\, H^2(\theta,\lambda)}{\gamma(r_{ell},\theta)} \tag{70}$$

where

- G denotes the Newtonian gravitational constant,
  $G = 6.67259 \cdot 10^{-11}\ \mathrm{m}^3 \cdot \mathrm{kg}^{-1} \cdot \mathrm{s}^{-2}$
- ρ denotes the density of the crust, $\rho = 2670\ \mathrm{kg} \cdot \mathrm{m}^{-3}$
- $r_{ell} = r_{ell}(\theta)$ denotes the radius-coordinate of point Q_0 on the reference ellipsoid

Height anomaly ell

$$\zeta_{ell}(r,\theta,\lambda) = \frac{T(r,\theta,\lambda)}{\gamma(r,\theta)} \tag{71}$$

Height anomaly

$$\zeta(r,\theta,\lambda) = \zeta_{ell}(r_{ell},\theta,\lambda) - \delta g_{sa}(r_{ell},\theta,\lambda)\,\frac{H(\theta,\lambda) + N(\theta,\lambda)}{\gamma(r_{ell},\theta)} \tag{72}$$

References

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Petrovskaya, M.S., Vershkov, A.N., 2006. Non-singular expressions for the gravity gradients in the local north-oriented and orbital reference frames. Journal of Geodesy 80, 117-127. doi: 10.1029/2011JB008916.

A Modified non-singular expressions for the gravity gradients in the LNOF

In this appendix, we will demonstrate how we modified the non-singular expressions for the gravity gradients in the LNOF (Eq. (10) - Eq. (15) and Eq. (60) - Eq. (65)). As an example, let us mentioned only one particular expression for the element T_{xx}, which has the following form

$$\begin{aligned} T_{xx} =& \frac{GM}{r^3} \sum_{n=2}^{M} \left(\frac{R}{r}\right)^n \sum_{m=0}^{n} \left(\overline{C}_{nm} \cos m\lambda + \overline{S}_{nm} \sin m\lambda\right) \\ & \times \Big(a_{nm}\, \overline{P}_{n,m-2}(\cos\theta) \\ & + \left[b_{nm} - (n+1)(n+2)\right] \overline{P}_{nm}(\cos\theta) \\ & + c_{nm}\, \overline{P}_{n,m+2}(\cos\theta)\Big), \end{aligned} \tag{73}$$

in which

$$a_{nm} = 0, \quad m = 0, 1 \tag{74}$$

$$\begin{aligned} a_{nm} =& \frac{\sqrt{1+\delta_{m,2}}}{4} \sqrt{n^2 - (m-1)^2} \\ & \times \sqrt{n+m}\sqrt{n-m+2}, \quad 2 \le m \le n \end{aligned} \tag{75}$$

$$b_{nm} = \frac{(n+m+1)(n+m+2)}{2\,(m+1)}, \quad m = 0, 1 \tag{76}$$

$$b_{nm} = \frac{n^2 + m^2 + 3n + 2}{2}, \quad 2 \le m \le n \tag{77}$$

$$\begin{aligned} c_{nm} =& \frac{\sqrt{1+\delta_{m,0}}}{4} \sqrt{n^2 - (m+1)^2}\sqrt{n-m} \\ & \times \sqrt{n+m+2}, \quad m = 0, 1 \end{aligned} \tag{78}$$

$$c_{nm} = \frac{1}{4}\sqrt{n^2 - (m+1)^2}\sqrt{n-m}\sqrt{n+m+2}, \quad 2 \le m \le n \tag{79}$$

$$\delta_{p,q} = \begin{cases} 1, & p = q, \\ 0, & p \ne q. \end{cases} \tag{80}$$

From Eq. (73), one can see that in addition to the term $\overline{P}_{nm}(\cos\theta)$, two other terms $\overline{P}_{n,m-2}(\cos\theta)$ and $\overline{P}_{n,m+2}(\cos\theta)$ must be computed for each m. From the practical numerical point of view, this is not an issue, if fixed-degree recursions have been used to evaluate fnALFs. In this case, for each m these terms have been already computed with the term $\overline{P}_{nm}(\cos\theta)$ essentially. In the GrafLab, however, we used fixed-order recursions, which are more frequently used in geodesy. In this case, with every change of m in the order-dependent loop, it is necessary to evaluate not only the term $\overline{P}_{nm}(\cos\theta)$, but also the two other terms. In other words, redundant computations occur. Thus, we modified Eq. (73) in the way that only the term $\overline{P}_{nm}(\cos\theta)$ is needed to be computed. We present Eq. (73) in the following form

$$
\begin{aligned}
T_{xx} =& \frac{GM}{r^3} \sum_{n=2}^{M} \left(\frac{R}{r}\right)^n \\
& \times \sum_{m=0}^{n} \Bigg[\left(\overline{C}_{n,m+2} \cos(m+2)\lambda + \overline{S}_{n,m+2} \sin(m+2)\lambda\right) a_{n,m+2} \\
& + \left(\overline{C}_{nm} \cos m\lambda + \overline{S}_{nm} \sin m\lambda\right) \left(b_{nm} - (n+1)(n+2)\right) \\
& + \left(\overline{C}_{n,m-2} \cos(m-2)\lambda + \overline{S}_{n,m-2} \sin(m-2)\lambda\right) c_{n,m-2} \Bigg] \\
& \times \overline{P}_{nm}(\cos\theta),
\end{aligned}
\tag{81}
$$

where

$$
\left.
\begin{array}{l}
\overline{C}_{n,m+2} \\
\overline{S}_{n,m+2} \\
\cos(m+2)\lambda \\
\sin(m+2)\lambda \\
a_{n,m+2}
\end{array}
\right\} = 0, \quad m+2 > n,
\tag{82}
$$

$$
\left.
\begin{array}{l}
\overline{C}_{n,m-2} \\
\overline{S}_{n,m-2} \\
\cos(m-2)\lambda \\
\sin(m-2)\lambda \\
c_{n,m-2}
\end{array}
\right\} = 0, \quad m-2 < 0.
\tag{83}
$$

The main idea of Eq. (81) is that the set of spherical harmonic coefficients is usually stored during the whole computational process, hence the coefficients $\overline{C}_{n,m+2}, \overline{S}_{n,m+2}$ and $\overline{C}_{n,m-2}, \overline{S}_{n,m-2}$ may be simply restored when necessary instead of the redundant computation of $\overline{P}_{n,m-2}(\cos\theta)$ and $\overline{P}_{n,m+2}(\cos\theta)$ in Eq. (73). The formulae for the remaining elements $T_{yy}, T_{zz}, T_{xy}, T_{xz}, T_{yz}$ may be easily modified in the same way.