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Rheological Properties of Temperate Firn

By Walter Ambach and Heinrich Eisner*

Summary: On an approximately 20 m deep firn pit in the accumulation area of an Alpine glacier, deformation measurements have been carried out over a period of 11 years. Evaluation of the data was performed by application of a Newtonian model, determining the shear- and bulk viscosity as well as by introduction of a non-linear constitutive equation for temperate firn. For the transition for firn into glacier ice, Glen's Flow Law for incompressible ice results.

Zusammenfassung: In einem etwa 20 m tiefen Firnschacht im Akkumulationsgebiet eines Alpengletschers wurden über 11 Jahre Verformungsmessungen durchgeführt. Die Auswertung erfolgte einerseits durch Anwendung eines Newton'schen Modells mit Bestimmung der Scher- und Volumsviskosität, andererseits durch Einführung eines nichtlinearen Fließgesetzes für temperierten Firn. Bei der Umwandlung von Firn in Gletschereis ergibt sich daraus das Glen'sche Fließgesetz für inkompressibles Eis.

1. INTRODUCTION

Temperate firn is snow with high density which has outlasted a balance year and may later turn into glacier ice by metamorphosis and refreezing of meltwater. The delimitation between firn and glacier ice is given by the fact that firn is an air- and water-permeable material, whereas glacier ice is air- and water-impermeable. In addition, firn is compressible, whereas glacier ice is treated as incompressible material. In a temperate glacier, the transition from firn to ice takes place in a depth of approximately 20 to 30 m, largely depending on the annual net balance.

In order to investigate the rheological properties of firn in a temperate glacier, deformation measurements of a firn pit were carried out between 1967 and 1978. Originally, the pit was 20 m deep and had a circular cross-section. In 14 different depths along the wall of the pit, 6 to 7 markers each were placed and their relative distances measured in intervals of one year. The deformed cross-sections of the pit were approximated by means of ellipses, the centres of the ellipses being located on the pit-axis. The tilt of the pit-axis was determined from the horizontal distances of the centres of the ellipses from the plumb line and from their relative vertical distances.

The pit is located in the central region of the accumulation area of Kesselwandferner (Oetztal Alps) in an altitude of 3240 m a.s.l. The water equivalent of the averaged annual net balance between 1967 and 1980 amounts to 1,3 m at this site. From velocity measurements on the surface it is known that longitudinal and transverse strain rates occur at the site of measurements (SCHNEIDER, 1970). With respect to the state of stress, the pit is therefore not located in a neutral zone.

Fig. 1 gives a schematic representation of the pit deformation for the period from 1967 to 1978. The measured values are the tilt of the pit-axis, the compression of the individual layers, the increase of the diameter in flow direction, and the decrease of the diameter in transverse direction. Moreover, the depth profile of density and the surface tilt along the flow distance are known (EISNER & AMBACH, 1981; SCHNEIDER, 1970). For the analysis, however, measured values for the period from 1967 to 1974 only were used, since from 1975 onwards, some of the measured values have been systematically disturbed by the formation of a large firn crevasse (EISNER et al., 1984a).

The lengths of the major and minor axes of the elliptically deformed cross-sections of the pit and the

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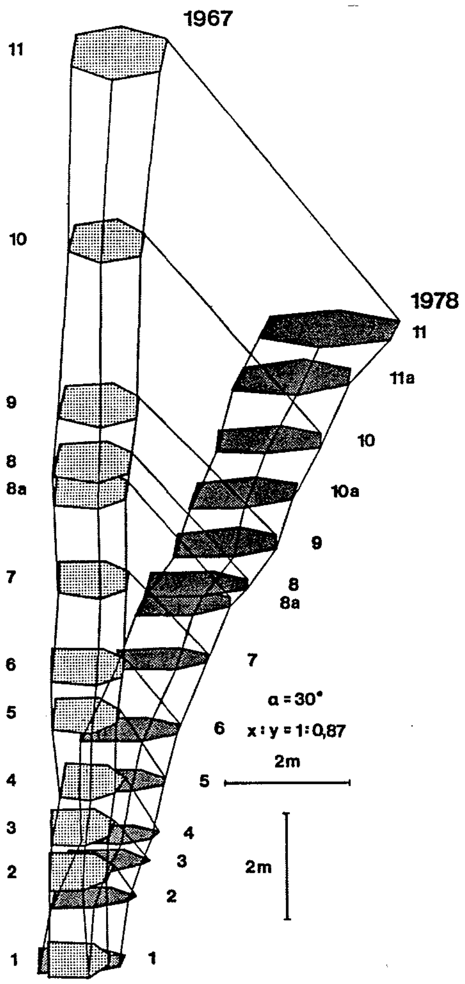

Fig. 1: Schematic representation of pit deformation for the period from 1967 to 1978 (From: EISNER & AMBACH, 1981).

Abb. 1: Schematische Darstellung der Schachtverformung in der Periode von 1967 bis 1978 (Aus: EISNER & AMBACH 1981).

thicknesses of the individual layers of firn were determined as functions of time. These functions can be approximated by straight lines, so that a strain rate results which is constant with respect to time (AMBACH & EISNER, 1986). The vertical course of the pit-axis was also approximated by a straight line.

The rheological properties of temperate firn can be treated from 2 different points of view:

- From point of view of linear snow mechanics:
  The shear viscosity μ and bulk viscosity η, being the material properties for the given state of stress, are being determined as "effective quantities". The analysis is based on the assumption of a Newtonian Model and represents a linear set-up.
- From point of view of non-linear ice mechanics:
  A non-linear constitutive equation for temperate firn is formulated, resulting in "Glen's Flow Law" at the transition from firn to ice.

2. NEWTON'S MODEL

In snow mechanics, deformations are often dealt with by means of a linear model (Newton's Model). It is being assumed that μ and η are functions of density and structure, but do not depend on the state of stress. This assumption is not correct (SALM, 1967), so that the results apply for the in situ state of stress only and have to be interpreted as "effective quantities". The linear model allows a multiaxial state of stress to be represented as linear superposition of uniaxial states of stress (Fig. 2), when the same values of μ and η apply for the multi-axial and the uni-axial state of stress.

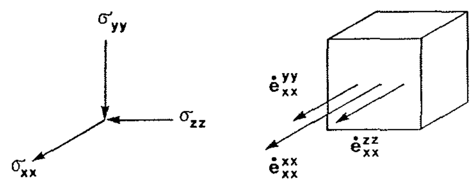

Fig. 2: Strain rate shares in x-direction, related to the indicated stresses σ_{xx}, σ_{yy}, σ_{zz}

Abb. 2: Strain rate-Werte in x-Richtung durch die angegebenen Spannungen σ_{xx}, σ_{yy}, σ_{zz}

The application of a linear model is required, since only the snow load is known from the state of stress at the pit and therefore the strain rates, caused by the snow load alone, can be related to it. The following linear constitutive equation is being introduced (EISNER et al., 1984b):

$$\sigma_{ij} = 2\mu\dot{e}_{ij} + (\eta - 2/3\mu)J_1\,\delta_{ij} \tag{1}$$

$$2\mu = \frac{\sigma'_{ij}}{\dot{e}'_{ij}}, \quad 3\eta = I_1/J_1 \tag{2}$$

with σ_{ij}, $\dot{e}_{ij}$ being the components of the stress tensor and the strain rate tensor, σ'_{ij}, $\dot{e}'_{ij}$ the deviators, I_1, J_1 the first invariants of the stress and the strain rate tensor, δ_{ij} Kronecker's Symbol, and μ, η the shear-and the bulk viscosity.

For the uni-axial state of stress, caused by the snow load σ_{yy}, one gets (EISNER at al., 1984b):

$$\mu = \frac{1}{2(1+\nu)} \cdot \frac{\sigma_{yy}}{\dot{e}^{yy}_{yy}} \tag{3}$$

$$\eta = 2/3\,\mu \cdot \frac{1+\nu}{1-2\nu} \tag{4}$$

with σ_{yy} being the stress resulting from the snow load, $\dot{e}^{yy}_{yy}$ the strain rate from the uni-axial state of stress σ_{yy} in y-direction, ν the viscous Poisson's Ratio, and μ, η the shear- and bulk viscosity. ν ist calculated as a function of density (BADER et al., 1951).

The values μ and η can be represented as a function of depth and density, however, they have to be interpreted as "effective quantities". In order to compare temperate firn with other types of snow, the "com-

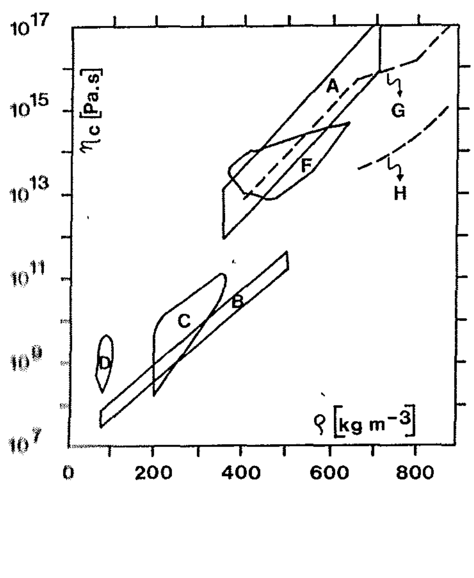

Fig. 3: Compactive viscosity η_c depending upon the density for various types of snow according to MELLOR (1975) (A-F), supplemented by values according to DÖRR & JESSBERGER (1983) (G) and by results of the present paper (H).
A: Greenland and Antarctica, —20 °C to —50 °C; B: Seasonal blanket of snow, Japan, 0 °C to —10 °C; C: Alps and Rocky Mountains; D: Creep tests with uni-axial deformation, —6 °C to —8 °C; F: Creep tests with uni-axial deformation, —23 °C to —48 °C; G: Antarctica (Georg von Neumayer-Station); H: Temperate firn (Oetztal Alps). (From: EISNER et al., 1984b).

Abb. 3: Compactive viscosity η_c in Abhängigkeit von der Dichte für verschiedene Schneesorten nach MELLOR (1975) (A-F), ergänzt mit Werten nach DÖRR & JESSBERGER (1983) (G) und den Ergebnissen der vorliegenden Arbeit (H).
A: Grönland und Antarktis, —20 °C bis —50 °C; B: Jahreszeitliche Schneedecke, Japan, 0 °C bis —10 °C; C: Alpen und Rocky Mountains; D: Kriechtests mit einachsiger Verformung, —6 °C bis —8 °C; F: Kriechtests mit einachsiger Verformung, —23 °C und —48 °C; G: Antarktis (Georg von Neumayer-Station); H: Temperierter Firn (Ötztaler Alpen).

pactive viscosity" η_c according to

$$\eta_c = \eta + 4/3\,\mu \qquad (5)$$

is calculated (MELLOR, 1975). The values of η_c fit with values of old snow, if extrapolated in larger density ranges (Fig. 3). They are, however, lower than the values of polar firn with the same density by a factor of approx. 10^{-2}, since the viscosity depends considerably upon the temperature.

3. NON-LINEAR CONSTITUTIVE EQUATION

The non-linear constitutive equation for temperate firn, developed here, consists of a deviatoric and an isotropic term and is expressed as

$$\dot{e}_{ij} = \dot{e}'_{ij} + 1/3\, J_1 \delta_{ij} \qquad (6)$$

In this equation, $\dot{e}_{ij}$ are the components of the strain rate tensor, $\dot{e}'_{ij}$ the components of the deviator, J_1 is the first invariant of the strain rate tensor, and δ_{ij} is Kronecker's Symbol. This set-up already contains the splitting-up of the strain rate components into a deviatoric term ($\dot{e}'_{ij}$) and an isotropic term ($1/3 J_1 \delta_{ij}$). The deviatoric term describes mere alterations of shape, caused by the stress deviator, whereas the isotropic term describes mere changes in volume, caused by a confining pressure. The confining pressure corresponds to the isotropic part of the tensor in the state of stress. At the transition from firn to ice, "Glen's Flow Law" for incompressible ice must result from the constitutive equation for temperate firn.

3.1 *The deviatoric term of the constitutive equation*

Expressed in components, "Glen's Flow Law" reads

$$\dot{e}'_{ii} = A\, \tau^2_{eff}\, \sigma'_{ij} \qquad (7)$$

A is a material constant depending upon temperature. All further symbols are defined in the list below. Analogously to equation (7), the equation.

$$\dot{e}'_{ij} = D(\varrho^*)\, \tau^2_{eff}\, \sigma'_{ij} \qquad (8)$$

is introduced, with ϱ^* being a dimensionless parameter for density as

$$\varrho^* = \frac{\varrho}{\varrho_{ICE} - \varrho} \qquad (9)$$

For ice, $\varrho^* \rightarrow \infty$ holds true. The factor $\tau^2_{eff}\sigma'_{ij}$ in equ. (8) describes the dependence $\dot{e}'_{ij}$ upon the stress, the function $D(\varrho^*)$ the dependence upon the density.

Fig. 4 shows the function $D(\varrho^*)$ with following characteristic properties:

- With increasing values of ϱ^*, $D(\varrho^*)$ decreases monotonously. This decrease corresponds to an increased resistance against alterations of shape of firn with higher densities.
- $D(\varrho^*)$ can be represented as sum of 2 exponential functions, covering both the left-hand steep range and the right-hand flat range in a satisfactory way.
- For large values of ϱ^*, $D(\varrho^*)$ approaches a constant value. To demonstrate this value graphically, the scale is extended by the factor 50 for $\varrho^* \geq 6$ in Fig. 4.

An adequate analytical shape for $D(\varrho^*)$ in the range of $2 \leq \varrho^* \leq 12$ reads

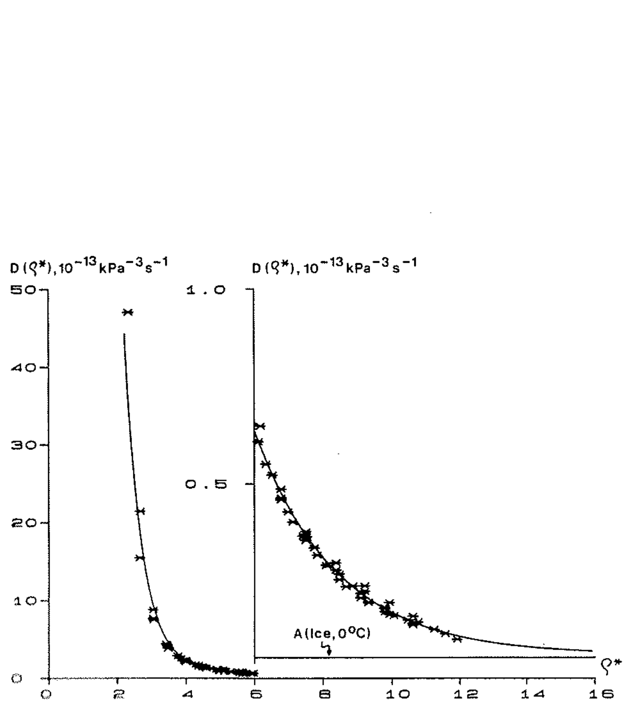

Fig. 4: Function D (ϱ^*) of the deviatoric term of the constitutive equation; Scale alteration for $\varrho^* \geq 6$.

Abb. 4: Funktion D (ϱ^*) des deviatorischen Terms des Fließgesetzes. Maßstabsänderung bei $\varrho^* = 6$.

$$D(\varrho^*) = A + D_1 \exp(-d_1 \varrho^*) + D_2 \exp(-d_2 \varrho^*) \tag{10}$$

The constants A, D_1, D_2, d_1, d_2 have been calculated numerically from the plot of the measured data by adaption of the parameters and are valid for the range of density from approx. 600 kg/m³ to 850 kg/m³.

3.2 Isotropic term of the constitutive equation

The isotropic term of the constitutive equation results, if J_1 is being substituted by I_1 in equ. 6. It then reads

$$J_1 = H(\varrho^*) I_1 \tag{11}$$

with J_1, I_1 being the first invariants of the strain rate tensor and the stress tensor. H (ϱ^*) is determined from the measured data by calculation of J_1/I_1. Fig. 5 shows that H (ϱ^*) for $2 \leq \varrho^* \leq 12$ can again be represented as sum of 2 exponential functions, i. e.

$$H(\varrho^*) = H_1 \exp(-h_1 \varrho^*) + H_2 \exp(-h_2 \varrho^*) \tag{12}$$

Here the constants H_1, H_2, h_1, h_2 are calculated numerically from the plot of the measured data by adap-

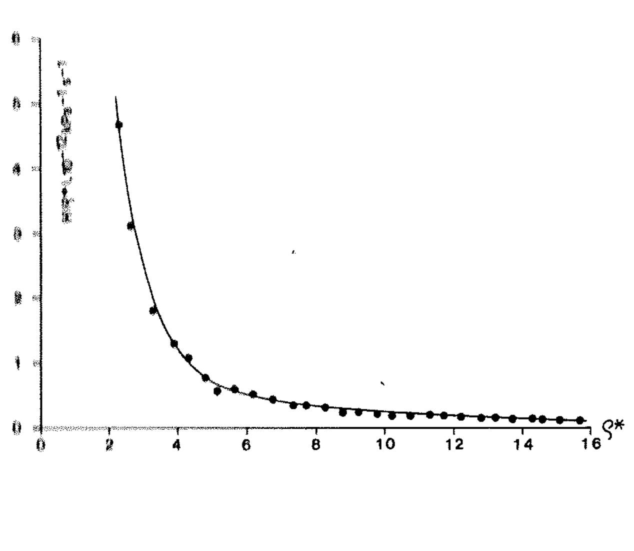

Fig. 5: Function H (ϱ^*) of the isotropic term of the constitutive equation.

Abb. 5: Funktion H (ϱ^*) des isotropen Terms des Fließgesetzes.

tion of the parameters. With increasing values for ϱ^*, $H(\varrho^*)$ decreases monotonously. This, again, corresponds to an increased resistance against changes in volume of firn with higher density. With $\varrho^* \rightarrow \infty$ one gets the limiting value $H(\varrho^*) \rightarrow 0$ for incompressible ice.

3.3 *Discussion and numerical results:*

An essential point of the analysis is the reconstruction of the components of the strain rate tensor and the stress tensor from the measured values. In this connection the following problems are of importance which are discussed comprehensively in the literature quoted:

- The calculation of the vertical stress component from the snow load by a profile of density, as well as the calculation of the resulting shear stress (AMBACH & EISNER, 1986). The cross-section profile of the glacier bed is taken into consideration according to NYE (1965).
- The transformation of the measured values for the pit deformation into corresponding deformation values for a solid body (EISNER et al., 1984b). The deformations have been measured at a cavity, whereas the constitutive equation is valid for a solid body.
- The dependency of the viscous Poisson's Ratio upon the density (BADER et al., 1951) and the state of stress (SALM, 1977). The viscous Poisson's Ratio is required for the above-mentioned transformation of the strain rate components of the cavity into those of a solid body. It is shown, however, that the viscous Poisson's Ratio does hardly influence the constitutive equation (AMBACH & EISNER, 1986).
- The calculation of the shear strain rate from the tilt rate of the pit axis (EISNER et al., 1984a).
- The set-up implies that in the constitutive equation, the same power is valid for snow and ice as far as the stress dependency of the deviatoric strain rate components is concerned. This power is assumed to be independent of the density of the firn (AMBACH & EISNER, 1986; MELLOR, 1975: 274, fig. 16).

The result is a non-linear constitutive equation for temperate firn in the range of $2 \leq \varrho^* \leq 12$:

$$\dot{e}_{ij} = D(\varrho^*)\tau^2_{eff}\sigma'_{ij} + 1/3\, H(\varrho^*)I_1\delta_{ij} \qquad (13)$$

$$D(\varrho^*) = A + D_1\exp(-d_1\varrho^*) + D_2\exp(-d_2\varrho^*) \qquad (14)$$

$$H(\varrho^*) = H_1\exp(-h_1\varrho^*) + H_2\exp(-d_2\varrho^*) \qquad (15)$$

$A = 6{,}04 \times 10^{-15}\ kPa^{-3}s^{-1}$, $D_1 = 3{,}94 \times 10^{-10} kPa^{-3}s^{-1}$
$D_2 = 7{,}07 \times 10^{-13}\ kPa^{-3}s^{-1}$, $d_1 = 2{,}071$, $d_2 = 0{,}419$
$H_1 = 4{,}74 \times 10^{-11}\ kPa^{-1}s^{-1}$, $H_2 = 9{,}64 \times 10^{-13}\ kPa^{-1}s^{-1}$
$h_1 = 1{,}081$, $h_2 = 0{,}131$

d_1, d_2, h_1, h_2 are dimensionless values. At the transition from firn to ice, "Glen's Flow Law", expressed in components, is obtained for $\varrho^* \rightarrow \infty$ from equ. 13

$$\dot{e}'_{ij} = A\tau^2_{eff}\sigma'_{ij} \qquad (16)$$

The constant A for temperate ice has been determined numerically by various authors. Because of the influence of the water- and dirt contents values for A show a large scatter PATERSON (1981: 39, table 3.3) provides a survey of these values and recommends a mean value of $A = 5{,}3 \times 10^{-15} kPa^{-3}s^{-1}$ for ice at 0 °C. Compared with this value, the present investigation results in $A = 6{,}04 \times 10^{-15} kPa^{-3}s^{-1}$. In view of the range of scattering of these values in literature, a satisfactory numerical agreement is obtained. Therefore "Glen's Flow Law" turns out to be the limiting case for the presently developed constitutive equation for temperate firn.

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List of symbols

A	constant of material
D_1, D_2 d_1, d_2	constant of material
$D(\varrho^*)$	parameter of constitutive equation, depending on ϱ^*
$\dot{e}_{ij}$ $(\dot{e}'_{ij})$	components of strain rate tensor (deviator)
$\dot{e}^{xx}_{xx}, \dot{e}^{yy}_{yy}, \dot{e}^{zz}_{zz}$	uniaxial strain rate in x-direction (subscript), caused by σ_{xx} (superscript), analogously in y, z directions
H_1, H_2 h_1, h_2	constants of material
$H(\varrho^*)$	parameter of the constitutive equation, depending on ϱ^*
I_1	first invariant of stress tensor
J_1	first invariant of strain rate tensor
δ_{ij}	Kronecker's symbol
η (η_c)	bulk viscosity (compactive viscosity)
μ	shear viscosity
ν	viscous Poisson's ratio
ϱ, ϱ_{ice}, ϱ^*	density (ϱ_{ice} = density of ice, $\varrho^* = \frac{\varrho}{\varrho_{ice} - \varrho}$)
σ_{ij} (σ'_{ij})	components of stress tensor (deviator)
τ_{eff}	effective stress
	$3\tau^2_{eff} = \sigma'^2_{xx} + \sigma'^2_{yy} + \sigma'^2_{zz} + 2\tau^2_{xy} + 2\tau^2_{xz} + 2\tau^2_{yz}$

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