Convergence and accuracy of sea ice dynamics solvers

Martin Losch (AWI)
Annika Fuchs (AWI), Jean-François Lemieux (EC)
Sea ice thickness with MITgcm (JPL)
Review of 2D momentum equations

\[ m \frac{D\mathbf{u}}{Dt} = -mf\mathbf{k} \times \mathbf{u} + \tau_{air} + \tau_{ocean} - m\nabla \phi(0) + \mathbf{F}, \]  \hspace{1cm} (1)

\[ F_j = \partial_i \sigma_{ij} = \text{divergence of symmetric stress tensor of rank 2} \]

Viscous-Plastic (VP) constitutive law (rheology):

\[ \sigma_{ij} = 2\eta \dot{\epsilon}_{ij} + [\zeta - \eta] \dot{\epsilon}_{kk} \delta_{ij} - \frac{P}{2} \delta_{ij}. \]  \hspace{1cm} (2)

EVP equations (NOT intended to be a different rheology):

\[ \frac{1}{E} \frac{\partial \sigma_{ij}}{\partial t} + \frac{1}{2\eta} \sigma_{ij} + \frac{\eta - \zeta}{4\zeta \eta} \sigma_{kk} \delta_{ij} + \frac{P}{4\zeta} \delta_{ij} = \dot{\epsilon}_{ij}. \]  \hspace{1cm} (3)

\[ \dot{\epsilon}_{ij} = \frac{1}{2}(\partial_i u_j + \partial_j u_i) = \text{strain rates} \]

\[ P = P^* hc \cdot e^{-C^*(1-c)} \quad \zeta = \min\left(\frac{P}{2\Delta}, \zeta_{\text{max}}\right) \quad \eta = \frac{\zeta}{e^2} \]
Review of 2D momentum equations

\[ m \frac{Du}{Dt} = -m f k \times u + \tau_{air} + \tau_{ocean} - m \nabla \phi(0) + F, \quad (1) \]

\[ F_j = \partial_i \sigma_{ij} = \text{divergence of symmetric stress tensor of rank 2} \]

Viscous-Plastic (VP) constitutive law (rheology):

\[ \sigma_{ij} = 2\eta \dot{\epsilon}_{ij} + [\zeta - \eta] \dot{\epsilon}_{kk} \delta_{ij} - \frac{P}{2} \delta_{ij}. \quad (2) \]

EVP equations (NOT intended to be a different rheology):

\[ \frac{1}{E} \frac{\partial \sigma_{ij}}{\partial t} + \frac{1}{2\eta} \sigma_{ij} + \frac{\eta - \zeta}{4\zeta \eta} \sigma_{kk} \delta_{ij} + \frac{P}{4\zeta} \delta_{ij} = \dot{\epsilon}_{ij}. \quad (3) \]

\[ \dot{\epsilon}_{ij} = \frac{1}{2} (\partial_i u_j + \partial_j u_i) = \text{strain rates} \]

\[ P = P^* h c \cdot e^{-C^*(1-c)} \quad \zeta = \min \left( \frac{P}{2\Delta}, \zeta_{\text{max}} \right) \quad \eta = \frac{\zeta}{e^2} \]
Review of 2D momentum equations

\[
m \frac{D \mathbf{u}}{D t} = -m \mathbf{f} \mathbf{k} \times \mathbf{u} + \tau_{air} + \tau_{ocean} - m \nabla \phi(0) + \mathbf{F},
\]  

\(F_j = \partial_i \sigma_{ij} = \) divergence of symmetric stress tensor of rank 2

Viscous-Plastic (VP) constitutive law (rheology):

\[
\sigma_{ij} = 2\eta \dot{\varepsilon}_{ij} + [\zeta - \eta] \dot{\varepsilon}_{kk} \delta_{ij} - \frac{P}{2} \delta_{ij}.
\]  

EVP equations (NOT intended to be a different rheology):

\[
\frac{1}{E} \frac{\partial \sigma_{ij}}{\partial t} + \frac{1}{2\eta} \sigma_{ij} + \frac{\eta - \zeta}{4\zeta \eta} \sigma_{kk} \delta_{ij} + \frac{P}{4\zeta} \delta_{ij} = \dot{\varepsilon}_{ij}.
\]  

\[
\dot{\varepsilon}_{ij} = \frac{1}{2} (\partial_i u_j + \partial_j u_i) = \text{strain rates}
\]

\[
P = P^* h c \cdot e^{-C^*(1-c)} \quad \zeta = \min \left( \frac{P}{2\Delta}, \zeta_{\text{max}} \right) \quad \eta = \frac{\zeta}{e^2}
\]
classical implicit VP-solver: the Picard solver

\[ A(x^{k-1}) \cdot x^{k-1} - b(x^{k-1}) = F(x^{k-1}) \]

outer (non-linear) loop:

until (some criterion), solve \( A(x^{k-1}) \cdot x^k = b(x^{k-1}) \)

\[ ||(x^k)^2|| \]

\[ || F(x^k) || \]

Lemieux and Tremblay (2009)
classical implicit VP-solver: the Picard solver

for \( A(x) \ x = b(x) \)

\[ A(x^{k-1}) \ x^k - b(x^{k-1}) = F(x^{k-1}) \]

outer (non-linear) loop:

until (some criterion), solve \( A(x^{k-1}) \ x^k = b(x^{k-1}) \)

\( \text{MITgcm default} \)

\( \text{Normalized avg KE} \)

Lemieux and Tremblay (2009)
We recall that in the original VP model [Hibler, 1979,], the numerical scheme is based on a modified Euler time step and an SOR solver. In this case, the linearized equation is solved twice per time step, which is equivalent to 2 OL iterations with our approach. Figure 3b shows the difference between the velocity field after 2 OL iterations and the FC velocity field. Errors of the same order of magnitude as the mean drift are present in a large portion of the domain. Errors as large as 15 cm s\(^{-1}\) are found close to Svalbard and errors of 1–5 cm s\(^{-1}\) are present in large regions of the central Arctic.

The lack of convergence of the approximate solution is also present in the normalized states of stress shown in Figure 4b. Many states of stress are neither viscous nor plastic, and are unrealistic as they lie outside of the yield curve. Note that these results remain the same when this experiment is repeated with a stand-alone SOR solver (as used by Hibler [1979]).

Some authors have used the average (or total) kinetic energy (KE) of the ice pack to assess the convergence of the approximate solution [e.g., Ip, 1993; Zhang and Hibler, 1997; Lemieux et al., 2008]. Lemieux et al. [2008] found that when performing 10 OL iterations, the average KE of the approximate solutions (for all the January 1997 time steps) is always within 2% of the FC average KE. The average KE is given by

\[
\frac{1}{N} \sum_{i} \left( \frac{u_i^2 + v_i^2}{2} \right)
\]

where \(N\) is the number of ice-covered grid cells (concentration larger than 50%) and \(u\) and \(v\) are the components of velocity interpolated at the tracer point. For the time step investigated here (6 January 1997 0000 UT), Figure 5 shows the average KE, normalized by the FC value, as a function of the number of OL iterations. Figure 5 indicates that in only three OL iterations, the average KE is within 2% of the FC value. Note that for this specific time step, the average KE after 10 OL iterations is within 0.2% of the FC value. From Figure 5, one could conclude that the nonlinear approximate solution converges very efficiently. This is however not the case. Figure 3c shows the difference between the velocity field obtained after 10 OL iterations and the FC velocity field. Even though the average KE is within 0.2% of the FC value, Figure 3c indicates that large errors, of the same order of magnitude as the mean drift, are still present in the velocity field. Errors as large as 6 cm s\(^{-1}\) are found close to Svalbard and errors of 1–2 cm s\(^{-1}\) are present in large regions of the central Arctic. Again, the states of stress confirm that the approximate solution is not a perfect VP solution (Figure 4c).

The position of the states of stress in principal stress space (or stress-invariant space) is a metric often used by modelers [e.g., Ip, 1993; Zhang and Hibler, 1997; Hunke and Dukowicz, 2002] to confirm that an approximate solution Figure 3.
EVP isn’t any better (but faster)

reference

EVP, 120 sub-cycles

EVP, 1980 sub-cycles

Lemieux et al. (2012), shear and divergence (per day)
The model was run for 10 days (17–27 January 2002) with either the JFNK or EVP solver. The spatial resolution of the simulations was chosen such that the grids cells are at least 100 km away from the land. These arch-like deformations in the EVP approximate solution are similar to the ones obtained by Lemieux et al. (2012), shear and divergence (per day)

EVP isn’t any better (but faster)

reference

EVP, 120 sub-cycles

EVP, 1980 sub-cycles

Lemieux et al. (2012), shear and divergence (per day)
new to sea ice dynamics: the JFNK solver

\[
[A(x^{k-1}) x^k - b(x^{k-1}) = F(x^k)]
\]

(0 =) \(F(x^{k-1}+\Delta x^k) = F(x^{k-1}) + F'(x^{k-1}) \Delta x^k, \quad F' = J(acobian)\)

until \(\| F(x^k) \| < \text{tol} \), solve \(J(x^{k-1})\Delta x^k = -F(x^{k-1})\)

and update \(x^k = x^{k-1} + \Delta x^k\)

\[\text{Newton method} \quad \text{Lemieux et al. (2010)}\]
JFNK in the MITgcm

- FGMRES with preconditioner (LSR, Zhang and Hibler, 1997)
- exact Jacobian times vector possible by AD
- parallel code:
  - scalar products in FGMRES
  - restricted additive Schwarz (RAS) method in LSR
- vector code:
  - iterative preconditioner (LSR)
  - coloring (zebra) method

Losch et al. (2014)
Parallel performance of solvers (4km Arctic configuration)

- Time (s) on the y-axis
- Number of CPUs on the x-axis
- Speed up

Legend:
- JFNK DYNSOLVER
- JFNK FGMRES
- JFNK JACVEC
- JFNK PRECOND
- LSOR DYNSOLVER
- EVP DYNSOLVER
Does it matter?

after nearly 40 years of simulation: average of Oct, 1995

Picard with 2 OL iterations

JFNK with II\|F\|<10^{-4}

JFNK – Picard

shear deformation (per day)

thickess (m), conc. (%)

0.0
0.6
1.2
1.8
2.4
3.0
3.6

0.0001
0.0002
0.0005
0.001
0.002
0.005
0.01
0.02
0.05
0.1
0.1
0.002
0.005
0.01
0.02
0.05
0.1

-0.45
-0.30
-0.15
0.00
0.15
0.30
0.45
“Timing” of solvers
(Is JFNK really faster?)
Accurate solvers affect

- ice thickness distribution
- linear kinematic features
- computer time!!!
Summary

- traditional Picard solver converges slowly
- EVP solver does not converge at all (to VP)
- JFNK-solver is efficient but expensive
Summary

- traditional Picard solver converges slowly
- EVP solver does not converge at all (to VP)
- JFNK-solver is efficient but expensive

JFNK-solver is available for large-scale problems