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*Junichi Minagawa, Thorsten Upmann*

## **Impressum:**

CESifo Working Papers

ISSN 2364-1428 (electronic version)

Publisher and distributor: Munich Society for the Promotion of Economic Research - CESifo GmbH

The international platform of Ludwigs-Maximilians University's Center for Economic Studies and the ifo Institute

Poschingerstr. 5, 81679 Munich, Germany

Telephone +49 (0)89 2180-2740, Telefax +49 (0)89 2180-17845, email [office@cesifo.de](mailto:office@cesifo.de)

Editors: Clemens Fuest, Oliver Falck, Jasmin Gröschl

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# On the Generalised Anti-inverse Elasticity Rule: An Existence Result

## Abstract

We consider an optimal commodity taxation problem under a consumption target and prove the existence of an optimal solution for the problem. This optimal solution obeys taxation rules that are contrary to standard taxation rules such as the inverse-elasticity rule. We also verify the necessary and sufficient condition for the optimal solution to exhibit uniform pricing.

JEL-Codes: H210.

Keywords: anti-inverse elasticity rule, consumption target, existence of an optimal solution, optimal commodity taxation, uniform pricing.

*Junichi Minagawa*  
*Faculty of Economics*  
*Chuo University*  
*742-1 Higashinakano, Hachioji*  
*Japan – Tokyo 192-0393*  
*minagawa@tamacc.chuo-u.ac.jp*

*Thorsten Upmann*  
*Helmholtz-Institute for Functional Marine*  
*Biodiversity at the Olden University*  
*(HIFMB)*  
*Ammerländer Heerstraße 231*  
*Germany – 23129 Oldenburg*  
*Thorsten.Upmann@hifmb.de*

April 30, 2018

## 1. INTRODUCTION

We consider a government facing a given consumption target for a group of commodities (such as different types of fuel), and explore how it should set taxes and subsidies on the commodities to accomplish that target. Minagawa and Upmann (2018) formulated an optimal commodity taxation model under such a consumption target where non-compliance with the target is allowed: a government chooses the consumer prices for a group of commodities to maximise consumer welfare minus the deviation cost of missing the target for the total consumption of the commodities. In that paper, the authors obtained an unconventional taxation rule: the generalised *anti*-inverse elasticity result, saying that *higher* prices should be charged for commodities with high price elasticities of total demand. An intuition for this result is that in order to attain the consumption target, a more price elastic commodity requires a smaller price change than does a less price elastic commodity. In this way, the target level is attained by relatively small price distortions and hence in a more efficient way.

However, the taxation rule is derived from the first-order conditions, implicitly assuming that the second-order conditions (or sufficient conditions) for optimality hold. Hence, the question of whether or not the taxation rule determined by the first-order conditions is indeed optimal remains open. This question of the optimality of the first-order taxation rules frequently arises in optimal taxation models where the objective function is not concave in the choice variables, as discussed by Mirrlees (1986, Sec. 2), and also cautioned by Myles (1995, pp. 113–14). But in the case considered by Minagawa and Upmann (2018) this issue is particularly significant as their result is contrary to standard taxation rules, which might raise suspicion on the optimality of that first order taxation rule.

In this paper, we address this problem of the validity of the first-order taxation rules. To this end, we consider the following optimal commodity taxation model under a consumption target: a government chooses the consumer prices for a group of commodities to maximise consumer welfare subject to the constraint that the total consumption of the commodities must meet a given target. Since under standard assumptions on preferences indirect utility functions are quasi-convex in prices, sufficient conditions for optimality are hard to verify in that case, as noted by Dixit (1990, p. 84). To deal with this difficulty, we choose an other route: we first demonstrate that there exists a solution to our problem; it then follows that, under a constraint qualification, the solution must satisfy the first-order conditions, and hence it obeys the resulting first-order taxation rule.

We next show that the first-order conditions derived here have the same form as those in the model of Minagawa and Upmann (2018). Thus, the generalised anti-inverse elasticity result mentioned above applies here as well, and it is indeed optimal. We also

prove the uniform pricing result that the optimal consumer prices are all equal if, and only if, the elasticities of Hicksian demand of the taxed commodity with respect to an untaxed commodity are all equal and non-negative (i. e., weakly substitutable); under homothetic preferences, this elasticity condition is equivalent to the condition that the elasticities of Marshallian demand of the taxed commodity with respect to the untaxed commodity are all equal. Finally, we provide an example that yields a unique optimal solution with uniform pricing.

## 2. MODEL

Consider the standard consumer model with one untaxed commodity (commodity 0), the quantity of which we denote by  $x_0 \geq 0$ , and  $n$  taxed commodities of a specific group, with an associated quantities  $\mathbf{x} \equiv (x_1, x_2, \dots, x_n) \in \mathbb{R}_+^n$ . Suppose that a consumer's preference relation is represented by a continuous utility function:  $u : \mathbb{R}_+^{n+1} \rightarrow \mathbb{R} : (x_0, \mathbf{x}) \mapsto u(x_0, \mathbf{x})$  satisfying:<sup>1</sup>

**Assumption 1.** *The utility function  $u$  is strictly increasing and strictly quasi-concave on  $\mathbb{R}_{++}^{n+1}$ , with  $u(x_0, \mathbf{x}) = c$  for any  $(x_0, \mathbf{x}) \in \mathbb{R}_+^{n+1} \setminus \mathbb{R}_{++}^{n+1}$ , for some  $c \in \mathbb{R}$ , and with  $u(x_0, \mathbf{x}) > c$  for any  $(x_0, \mathbf{x}) \in \mathbb{R}_{++}^{n+1}$ .*

Let  $q_0 > 0$  and  $\mathbf{q} \equiv (q_1, q_2, \dots, q_n) \in \mathbb{R}_+^n$  denote the consumer prices of commodities 0 and 1,  $\dots$ ,  $n$  respectively. Hence, the total cost of consumption of the taxed commodities amounts to  $\mathbf{q} \cdot \mathbf{x} \equiv \sum_{i=1}^n q_i x_i$ . Let  $I > 0$  denote the consumer's income. Then, the consumer solves:

$$\underset{x_0, \mathbf{x}}{\text{Maximise}} \quad u(x_0, \mathbf{x}) \quad \text{s. t.} \quad q_0 x_0 + \mathbf{q} \cdot \mathbf{x} \leq I, \quad (1)$$

which yields, for any  $(q_0, \mathbf{q}, I)$ , a unique interior solution. This solution is represented by continuous Marshallian demand functions,  $x_0^m : \mathbb{R}_{++}^{n+2} \rightarrow \mathbb{R}_{++} : (q_0, \mathbf{q}, I) \mapsto x_0^m(q_0, \mathbf{q}, I)$  and  $\mathbf{x}^m : \mathbb{R}_{++}^{n+2} \rightarrow \mathbb{R}_+^n : (q_0, \mathbf{q}, I) \mapsto \mathbf{x}^m(q_0, \mathbf{q}, I)$ . Correspondingly, Hicksian (or compensated) demand functions are written by  $x_0^h(q_0, \mathbf{q}, v)$  and  $\mathbf{x}^h(q_0, \mathbf{q}, v)$ , respectively, where  $v$  represents a given utility level.

Let  $p_0 > 0$  and  $\mathbf{p} \equiv (p_1, p_2, \dots, p_n) \in \mathbb{R}_+^n$  denote the fixed net prices of commodities 0 and 1,  $\dots$ ,  $n$  respectively. Let  $\mathbf{t} \equiv (t_1, t_2, \dots, t_n) \in \mathbb{R}^n$  denote the unit taxes imposed on commodities 1,  $\dots$ ,  $n$ . The consumer prices are thus written by  $q_0 \equiv p_0$  and  $\mathbf{q} \equiv \mathbf{p} + \mathbf{t}$ , and the government's choice variables are the consumer prices  $\mathbf{q}$  (or the unit taxes  $\mathbf{t}$ ). Let the government's objective function be the indirect utility function defined by  $V(q_0, \mathbf{q}, I) \equiv u(x_0^m(q_0, \mathbf{q}, I), \mathbf{x}^m(q_0, \mathbf{q}, I))$ . Define the Marshallian total demand function for the  $n$  taxed

<sup>1</sup>We say that a function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  is strictly increasing on  $S \subset \mathbb{R}^n$  whenever for any  $a, b \in S$  such that  $a_i \geq b_i$  for all  $i$  and  $a \neq b$ ,  $f(a) > f(b)$ .

commodities by  $X^m(q_0, \mathbf{q}, I) \equiv \sum_{i=1}^n x_i^m(q_0, \mathbf{q}, I)$ . Then, the optimal commodity taxation problem of the government facing a quantity constraint is:

$$\underset{\mathbf{q}}{\text{Maximise}} V(q_0, \mathbf{q}, I) \quad \text{s. t.} \quad X^m(q_0, \mathbf{q}, I) = Z, \quad (2)$$

where  $Z > 0$  represents a given target level of total consumption of  $\mathbf{x}$ .<sup>2</sup>

### 3. RESULTS

In order to establish the existence of a solution to problem (2), we make the following assumption on the feasibility of the consumption target.<sup>3</sup>

**Assumption 2.** *For any  $Z$  there is a price vector  $\mathbf{q}$  such that  $X^m(q_0, \mathbf{q}, I) = Z$ .*

We now prove the following existence result.

**Proposition 1.** *Under Assumptions 1 and 2, there exists a solution to problem (2).*

**Proof.**<sup>4</sup> We first show that there exists a solution to the auxiliary problem

$$\underset{\mathbf{q}}{\text{Maximise}} V(q_0, \mathbf{q}, I) \quad \text{s. t.} \quad X^m(q_0, \mathbf{q}, I) = Z, \quad q_j \geq \varepsilon_Z, \quad j = 1, 2, \dots, n, \quad (*)$$

for some suitably small  $\varepsilon_Z > 0$ . It follows from Assumption 2 that for any given value  $Z$ , there exists some  $\varepsilon_Z > 0$  such that the set  $Q_{Z, \varepsilon_Z} \equiv \{\mathbf{q} \in \mathbb{R}_{++}^n \mid X^m(q_0, \mathbf{q}, I) = Z, q_j \geq \varepsilon_Z, j = 1, 2, \dots, n\}$  is non-empty. That is, we can find a price vector  $\bar{\mathbf{q}} \in Q_{Z, \varepsilon_Z}$  with  $x_i^m(q_0, \bar{\mathbf{q}}, I) > 0$  for all  $i = 0, 1, \dots, n$ . Let  $\bar{V} \equiv V(q_0, \bar{\mathbf{q}}, I)$ , which is greater than  $c \in \mathbb{R}$  by Assumption 1. We then define the set  $Q_{\bar{V}} \equiv \{\mathbf{q} \in \mathbb{R}_{++}^n \mid V(q_0, \mathbf{q}, I) \geq \bar{V}\}$ . Let  $Q \equiv Q_{Z, \varepsilon_Z} \cap Q_{\bar{V}}$ . Since  $\bar{\mathbf{q}} \in Q$ , the set  $Q$  is non-empty.

In the following, we will show that the set  $Q$  is compact. First, we prove that the set  $Q$  is closed. Let  $Q_Z \equiv \{\mathbf{q} \in \mathbb{R}_{++}^n \mid X^m(q_0, \mathbf{q}, I) = Z\}$ . By the continuity of the Marshallian demand functions, the set  $Q_Z$  is closed in  $\mathbb{R}_{++}^n$ . Similarly, by the continuity of the indirect

<sup>2</sup>Some applications require a greater-than-equal-to constraint, demanding that total consumption of  $n$  commodities may not fall short of some minimum level (e. g., merit goods), while other applications require a less-than-equal-to constraint, demanding that total consumption may not exceed some maximum level (e. g., demerit goods). Since in either case a *binding* constraint becomes an equality constraint, we consider that case here.

<sup>3</sup>Since the case of a single taxed commodity is trivial—in fact, it boils down to the standard textbook model with two goods: an untaxed good and a taxed good—we are interested in the case of two or more taxed commodities (i. e.,  $n \geq 2$ ). In this case, there are generically many price vectors leading to the same level of total consumption (see Figure 1 for a case of two taxed commodities).

<sup>4</sup>The idea of this proof is borrowed from Iritani (1986, Sec. 1.6), who shows the existence of a solution for the standard optimal commodity tax problem with a revenue constraint.

utility function, the set  $Q_{\bar{V}}$  is closed in  $\mathbb{R}_{++}^n$ . Then the set  $Q_Z \cap Q_{\bar{V}}$  is closed in  $\mathbb{R}_{++}^n$ . Let  $Q_{\varepsilon_Z} \equiv \{\mathbf{q} \in \mathbb{R}_{++}^n \mid q_j \geq \varepsilon_Z, j = 1, 2, \dots, n\}$ . The set  $Q_{\varepsilon_Z}$  is a subset of  $\mathbb{R}_{++}^n$  and is closed (in  $\mathbb{R}^n$ ). The set  $Q \equiv Q_Z \cap Q_{\bar{V}} \cap Q_{\varepsilon_Z}$  is therefore closed (in  $\mathbb{R}^n$ ).

Second, we prove that the set  $Q$  is bounded. By way of contradiction, suppose not. Then, there exists a sequence of prices  $\{\mathbf{q}^\nu\}_{\nu=1}^\infty$  in  $Q$  with  $\|\mathbf{q}^\nu\| \rightarrow \infty$ . Now, for each  $\nu$ , let  $r_0^\nu \equiv q_0/(q_0 + \sum_{i=1}^n q_i^\nu)$ ,  $r_j^\nu \equiv q_j^\nu/(q_0 + \sum_{i=1}^n q_i^\nu)$ ,  $j = 1, 2, \dots, n$ , and  $I^\nu \equiv I/(q_0 + \sum_{i=1}^n q_i^\nu)$ . By passing to a subsequence if necessary, we may assume that the sequence of prices and incomes  $\{(r_0^\nu, \mathbf{r}^\nu, I^\nu)\}_{\nu=1}^\infty$  is such that  $r_0^\nu \rightarrow 0$ ,  $\mathbf{r}^\nu \rightarrow \mathbf{r}'$ , and  $I^\nu \rightarrow 0$  where each element of  $\mathbf{r}'$  is in  $[0, 1]$  such that  $r'_k$  is non-zero for some commodity  $k$ .<sup>5</sup>

Then, from the homogeneity of degree zero, we have for each  $\nu$ ,  $x_i^m(q_0, \mathbf{q}^\nu, I) = x_i^m(r_0^\nu, \mathbf{r}^\nu, I^\nu)$  for all  $i = 0, 1, \dots, n$ . Moreover, it follows that  $x_k^m(r_0^\nu, \mathbf{r}^\nu, I^\nu) \rightarrow 0$ , since, by the budget constraint, we have  $0 \leq x_k^m(r_0^\nu, \mathbf{r}^\nu, I^\nu) \leq I^\nu/r_k^\nu$  for each  $\nu$ . But then, since for each  $\nu$ ,  $V(q_0, \mathbf{q}^\nu, I) = V(r_0^\nu, \mathbf{r}^\nu, I^\nu) \geq \bar{V} > c$ , Assumption 1 implies that there is some commodity  $l$  such that  $x_l^m(r_0^\nu, \mathbf{r}^\nu, I^\nu) \rightarrow \infty$  (since otherwise,  $V(r_0^\nu, \mathbf{r}^\nu, I^\nu) \rightarrow c$ ). This implies, together with the quantity constraint  $X^m(r_0^\nu, \mathbf{r}^\nu, I^\nu) = Z$  for each  $\nu$ , that  $x_0^m(r_0^\nu, \mathbf{r}^\nu, I^\nu) \rightarrow \infty$ . On the other hand, by the budget constraint, we have for each  $\nu$ ,  $r_0^\nu x_0^m(r_0^\nu, \mathbf{r}^\nu, I^\nu) + \mathbf{r}^\nu \cdot \mathbf{x}^m(r_0^\nu, \mathbf{r}^\nu, I^\nu) = I^\nu$ , which is equal to  $q_0 x_0^m(r_0^\nu, \mathbf{r}^\nu, I^\nu) + \mathbf{q}^\nu \cdot \mathbf{x}^m(r_0^\nu, \mathbf{r}^\nu, I^\nu) = I$ . It thus follows that for each  $\nu$ ,  $x_0^m(r_0^\nu, \mathbf{r}^\nu, I^\nu) = [I - \mathbf{q}^\nu \cdot \mathbf{x}^m(r_0^\nu, \mathbf{r}^\nu, I^\nu)]/q_0$ . Since the right-hand side is bounded above, we obtain a contradiction. Hence, the set  $Q$  is bounded. The set  $Q$  is thus non-empty and compact, and the indirect utility function is continuous. Therefore, by Weierstrass' theorem, there exists a solution of problem (\*).

We next establish the existence of a solution of problem (2). Consider the sequence  $\{\varepsilon^\nu\}_{\nu=N+1}^\infty$  where  $\varepsilon^\nu \equiv 1/\nu$ , and thus  $\varepsilon^\nu \rightarrow 0$ . Let  $N$  be a sufficiently large integer. Then, for each  $\nu$ , there exists a solution of problem (\*) with  $\varepsilon_Z \equiv \varepsilon^\nu$ , and hence we may denote it by  $\mathbf{q}_\varepsilon^\nu$ . We will prove that the sequence  $\{\mathbf{q}_\varepsilon^\nu\}_{\nu=N+1}^\infty$  has an accumulation point  $\mathbf{q}^*$  in  $\mathbb{R}_{++}^n$ , that is,  $\{\mathbf{q}_\varepsilon^\nu\}_{\nu=N+1}^\infty$  has some subsequence that converges to  $\mathbf{q}^* \in \mathbb{R}_{++}^n$ . By way of contradiction, suppose not. Then, only two cases are possible: (i)  $\{\mathbf{q}_\varepsilon^\nu\}_{\nu=N+1}^\infty$  has no convergent subsequence in  $\mathbb{R}_+^n$ ; thus,  $\|\mathbf{q}_\varepsilon^\nu\| \rightarrow \infty$ , or (ii)  $\{\mathbf{q}_\varepsilon^\nu\}_{\nu=N+1}^\infty$  has some subsequence that converges to a point in  $\mathbb{R}_+^n \setminus \mathbb{R}_{++}^n$ . In case (i), by similar arguments to the above, we can derive a contradiction. In case (ii), it can be proved as in Theorem 1.3.9 of Aliprantis, Brown, Burkinshaw (1990) that there is some commodity  $l$  such that  $x_l^m(q_0, \mathbf{q}_\varepsilon^\nu, I) \rightarrow \infty$ . Thus, again, by similar arguments to those above, we obtain a contradiction. Therefore, the sequence  $\{\mathbf{q}_\varepsilon^\nu\}_{\nu=N+1}^\infty$  has an accumulation point  $\mathbf{q}^*$  in  $\mathbb{R}_{++}^n$ . Clearly,  $\mathbf{q}^*$  is a solution of problem (2).  $\square$

<sup>5</sup>Note that  $r'_k$  does not need to be equal to 1. For example, consider  $r_k^\nu = q_k^\nu/(q_0 + \sum_{i=1}^n q_i^\nu)$ . If  $q_1^\nu = \dots = q_n^\nu$  for each  $\nu$ , then  $r_k^\nu \rightarrow 1/n$ .

The situation is illustrated in Figure 1. The figure displays four different indifference curves, the curve of constant total consumption passing through  $\bar{q}$  and  $\mathbf{q}^*$ , and the shaded area representing the set  $Q_{\bar{V}}$ . Then, given the level of total consumption  $Z = x_1 + x_2$ , the utility maximising price vector equals  $\mathbf{q}^*$ . (In our example provided below a closed-form solution for  $\mathbf{q}^*$  is available.)

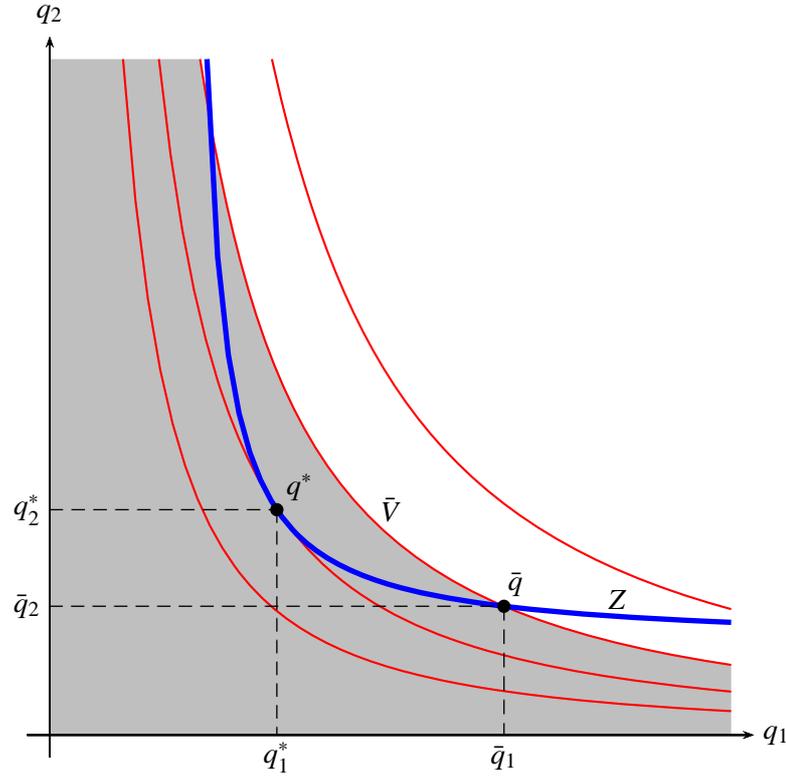


FIGURE 1. Curves of constant utility (red, thin) and constant total consumption (blue, thick)

In the following, we assume that the utility function and the demand functions are continuously differentiable. Let  $\varepsilon_{ij}^m \equiv (\partial x_i^m / \partial q_j)(q_j / x_i)$  represent the elasticity of Marshallian demand of commodity  $i$  with respect to the consumer price of commodity  $j$ ; and let  $\varepsilon_{ij}^h \equiv (q_j / x_i^h)(\partial x_i^h / \partial q_j)$  represent the corresponding elasticity of Hicksian demand. Moreover, we define the income share of commodity  $j$  by  $\eta_j \equiv q_j x_j^m / I$ . We simply write  $x_i$  to denote the level of the demand under consideration.

Suppose that at least one of the  $n$  derivatives  $(\partial / \partial q_j) X^m(q_0, \mathbf{q}, I)$ ,  $j = 1, 2, \dots, n$  at a solution for problem (2) is non-zero; then the usual constraint qualification, the so-called rank condition, is satisfied. Since by Proposition 1 there exists a solution for problem (2),  $\mathbf{q}^*$  in  $\mathbb{R}_{++}^n$ , the solution must satisfy the first order conditions. Using the Lagrangian for problem (2),  $L(\mathbf{q}, \lambda) \equiv V(q_0, \mathbf{q}, I) + \lambda[Z - X^m(q_0, \mathbf{q}, I)]$ , and applying Roy's identity to the

first order conditions, we obtain

$$-\mu x_j - \lambda \frac{\partial X^m}{\partial q_j} = 0, \quad j = 1, 2, \dots, n, \quad (3)$$

where  $\partial X^m / \partial q_j \equiv \sum_{i=1}^n (\partial x_i^m / \partial q_j)$ , and  $\mu \equiv \partial V / \partial I$  denotes the marginal utility of income.<sup>6</sup>

**Remark 1.** Equation (3) has the same form as equation (2) in Minagawa and Upmann (2018). Thus, the main taxation rule obtained there applies here as well. To see this, let  $\sigma_j^m \equiv (\partial X^m / \partial q_j)(q_j / X)$  be the elasticity of Marshallian total demand with respect to the price of taxed commodity  $j$ , and let  $v_j \equiv x_j / X$  be the demand share of that commodity. Then, we obtain from equation (3),

$$q_j = -\frac{\lambda \sigma_j^m}{\mu v_j}, \quad j = 1, 2, \dots, n. \quad (4)$$

This implies the generalised anti-inverse elasticity result that the solution of problem (2) is proportional to the price elasticity  $\sigma_j^m$  (but is inversely proportional to the consumption share  $v_j$ ).

Moreover, the following uniform pricing result holds as well:

**Proposition 2.** The optimal consumer prices of problem (2) are all equal if, and only if, the elasticities of Hicksian demand of the taxed commodity with respect to the untaxed commodity are all equal and non-negative (i. e., weakly substitutable):<sup>7</sup>

$$q_j = q, \quad \forall j \neq 0 \quad \Leftrightarrow \quad \varepsilon_{j0}^h = \alpha \geq 0, \quad \forall j \neq 0. \quad (5)$$

**Proof.** ( $\Rightarrow$ ) Using the Slutsky equation, we may express equation (3) as

$$-\frac{\mu}{\lambda} + \sum_{i=1}^n \frac{\partial x_i^m}{\partial I} = \frac{1}{x_j} \sum_{i=1}^n \frac{\partial x_i^h}{\partial q_j}, \quad j = 1, 2, \dots, n. \quad (6)$$

The left-hand side of equation (6) is independent of  $j$ , and we denote it by  $\theta$ . Then, using the fact that  $\partial x_i^h / \partial q_j = \partial x_j^h / \partial q_i$  and the compensated price elasticities, we may express equation (6) as

$$\theta = \sum_{i=1}^n \frac{\varepsilon_{ji}^h}{q_i}, \quad j = 1, 2, \dots, n. \quad (7)$$

<sup>6</sup>Likewise, we may interpret the multiplier  $\lambda$  as the marginal utility of public consumption, since  $\partial V(q_0, \mathbf{q}(Z), I) / \partial Z = \lambda$  where  $q_j = q_j(Z)$  is a solution of problem (2).

<sup>7</sup>This elasticity condition is the same as the necessary and sufficient condition for uniform taxation (explored by Diamond and Mirrlees 1971, Sandmo, 1974, and Sadka, 1977). That is, in the standard optimal commodity taxation model, the same tax rates should be imposed on commodities if, and only if, all commodities are equally weakly substitutable with respect to leisure.

For equal consumer prices,  $q_i = q \forall i = 1, \dots, n$ , the right-hand side of equation (7) becomes  $(1/q) \sum_{i=1}^n \varepsilon_{ji}^h$ . Using Hicks' "third law," we get  $(1/q) \sum_{i=1}^n \varepsilon_{ji}^h = (1/q)(-\varepsilon_{j0}^h)$ . Hence, all elasticities  $\varepsilon_{j0}^h$  ( $j \neq 0$ ) must be equal, say  $\varepsilon_{j0}^h = \alpha$  ( $j \neq 0$ ). Substituting this into the relation  $\sum_{j=0}^n \eta_j \varepsilon_{j0}^h = 0$ ,<sup>8</sup> together with the fact that  $\varepsilon_{00}^h \leq 0$ , we obtain  $\alpha \sum_{j=1}^n \eta_j = -\eta_0 \varepsilon_{00}^h \geq 0$ ; therefore,  $\alpha \geq 0$ .

( $\Leftarrow$ ) Substituting the relation  $\partial x_0^h / \partial q_j = \partial x_j^h / \partial q_0 = \alpha x_j / q_0$  into the Slutsky equation, we obtain  $\partial x_0^m / \partial q_j = \phi x_j / q_0$  where  $\phi \equiv \alpha - (\partial x_0^m / \partial I) q_0$  is independent of  $j$ . Differentiating both sides of the identity  $q_0 x_0^m(q_0, \mathbf{q}, I) + \mathbf{q} \cdot \mathbf{x}^m(q_0, \mathbf{q}, I) \equiv I$  with respect to  $q_j$  and then using  $\partial x_0^m / \partial q_j = \phi x_j / q_0$ , we get

$$x_j = -\frac{1}{1 + \phi} \sum_{i=1}^n q_i \frac{\partial x_i^m}{\partial q_j}. \quad (8)$$

Substituting equation (8) into equation (3), we obtain

$$\sum_{i=1}^n \left[ \frac{\mu}{(1 + \phi)} q_i - \lambda \right] \frac{\partial x_i^m}{\partial q_j} = 0, \quad j = 1, 2, \dots, n. \quad (9)$$

Hence,  $q_i = q \equiv (1 + \phi)\lambda/\mu$ ,  $\forall i \neq 0$ , is a solution of equation (9).<sup>9</sup>  $\square$

**Remark 2.** *With homothetic preferences, all income elasticities are equal to 1. It then follows from the Slutsky equation in elasticity form that  $\varepsilon_{j0}^h = \alpha$  ( $\geq 0$ ),  $\forall j \neq 0 \Leftrightarrow \varepsilon_{j0}^m = \beta \equiv \alpha - \eta_0$ ,  $\forall j \neq 0$ . The uniform pricing result in Proposition 2 may thus be written as:*

$$q_j = q, \quad \forall j \neq 0 \quad \Leftrightarrow \quad \varepsilon_{j0}^m = \beta \equiv \alpha - \eta_0, \quad \forall j \neq 0. \quad (10)$$

*That is, under homothetic preferences, the optimal consumer prices are all equal if, and only if, the elasticities of Marshallian demand of the taxed commodity with respect to the untaxed commodity are all equal.*

**Example.** Consider the consumer's problem with two taxed commodities. Suppose that the preference relation is represented by a Cobb–Douglas utility function  $u(x_0, \mathbf{x}) \equiv x_0 x_1 x_2$ . Let  $q_0 = 1$ . Solving problem (1), we have  $x_i^m(q_0, \mathbf{q}, I) = I/(3q_i)$ ,  $\forall i$ , and then  $V(q_0, \mathbf{q}, I) = I^3/(27q_1 q_2)$ . Next, consider problem (2). The first order conditions give rise to the unique solution  $q_1^* = q_2^* = 2I/(3Z)$ , which represents uniform-pricing since  $\varepsilon_{10}^m = \varepsilon_{20}^m = 0$ , and

<sup>8</sup>It follows from the Cournot aggregation, the Engel aggregation, and the Slutsky equation that  $\sum_{j=0}^n \eta_j \varepsilon_{jk}^h = 0$ ,  $k = 0, 1, \dots, n$ .

<sup>9</sup>If the matrix  $(\partial x_i^m / \partial q_j)$  ( $i, j = 1, 2, \dots, n$ ) is non-singular, then the solution is unique.

$\lambda^* = IZ/6$ . Moreover, the Hessian matrix of the Lagrangian at  $(q_1^*, q_2^*, \lambda^*)$  is

$$\begin{pmatrix} \frac{\partial^2 L}{\partial q_1^2} & \frac{\partial^2 L}{\partial q_1 \partial q_2} & \frac{\partial^2 L}{\partial q_1 \partial \lambda} \\ \frac{\partial^2 L}{\partial q_2 \partial q_1} & \frac{\partial^2 L}{\partial q_2^2} & \frac{\partial^2 L}{\partial q_2 \partial \lambda} \\ \frac{\partial^2 L}{\partial \lambda \partial q_1} & \frac{\partial^2 L}{\partial \lambda \partial q_2} & \frac{\partial^2 L}{\partial \lambda^2} \end{pmatrix} = \begin{pmatrix} 0 & \frac{3Z^4}{16I} & \frac{3Z^2}{4I} \\ \frac{3Z^4}{16I} & 0 & \frac{3Z^2}{4I} \\ \frac{3Z^2}{4I} & \frac{3Z^2}{4I} & 0 \end{pmatrix}. \quad (11)$$

Since the determinant of this matrix is  $27Z^8/(128I^3) > 0$ , the second order condition is satisfied.<sup>10</sup> Therefore, the price vector  $\mathbf{q}^* = (2I/(3Z), 2I/(3Z))$  is indeed optimal.

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<sup>10</sup>The second order condition for the constraint maximisation problem with two variables and one equality constraint is that the determinant of the Hessian matrix of the Lagrangian is positive. See, e. g., Silberberg and Suen (2001, Sec. 6.5).