A hybrid nonlinear-Kalman ensemble transform filter for data assimilation in systems with different degrees of nonlinearity

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PDAF Parallel Data Assimilation Framework

Overview

The second-order exact particle filter NETF (nonlinear ensemble transform filter) is combined with local ensemble transform Kalman filter (LETKF) to build a hybrid filter method. The filter combines the stability of the LETKF with the nonlinear properties of the NETF to obtain improved assimilation results for small ensemble sizes. Both filter components are localized in a consistent way so that the filter can be applied with high-dimensional models.

The degree of filter nonlinearity is defined by a hybrid weight which shifts the analysis between the LETKF and NETF. Since the NETF is more sensitive to sampling errors than the LETKF, the latter filter should be preferred in linear cases. Accordingly the adaptive hybrid weight is defined based on the nonlinearity of the system so that the adaptivity yields a good filter performance in linear and nonlinear situations.

Hybrid weight γ

Here, we define different rules to compute the hybrid weight γ adaptively.

Using the effective sample size $N_{eff} = \sum (w^i)^{-2}$ **:**

γ_α Choose γ_α so that N_{eff} is as small as possible, but above a limiting value α [see 3]

New alternative linear dependence

γ_{lin} $\gamma_{lin} = 1 - \beta N_{eff}/N$

Note: It is known that if N_{eff} is close to 1, particle filters don't work well. However, this does not imply that the PF is better then the LETKF for higher N_{eff}.

Using the skewness and kurtosis of the observed ensemble:

Kalman filters assume that distributions are Gaussian. In this case the LETKF is preferable. We use skewness & kurtosis to quantify the non-Gaussianity.

In general skewness (*skew*) and kurtosis (*kurt*) are not bounded. However, we can normalized the skewness and kurtosis by

$$skew' = skew/\sqrt{N} \qquad kurt' = kurt/N$$

We can define

$$\gamma_{sk} = \min(1 - |kurt'|, 1 - |skew'|)$$

To avoid too low N_{eff} we define combined rules

$\gamma_{sk,lin}$ $\gamma_{sk,lin} = max[\gamma_{sk},\ \gamma_{lin}]$

$\gamma_{sk,\alpha}$ $\gamma_{sk,\alpha} = max[\gamma_{sk},\ \gamma_\alpha]$

Linear and Nonlinear Filters

The transformation of the ensemble mean and ensemble perturbations for ensemble size N can be written in the generic form:

$$\overline{\mathbf{x}}^a = \overline{\mathbf{x}}^f + \mathbf{X}'^f \tilde{\mathbf{w}}$$
$$\mathbf{X}'^a = \mathbf{X}'^f \mathbf{W}$$

Ensemble Kalman & nonlinear filters just use different definitions of the

- weight vector $\tilde{\mathbf{w}}$ (dimension N)
- Transform matrix $\mathbf{W}$ (dimension $N \times N$)

NETF

NETF [1, 2] is a second-order exact particle filter. We compute the normalized weight vector $\mathbf{w} = (w^{(1)}, \ldots, w^{(N)})/\sum w^{(i)}$ using likelihood weights. For Gaussian observation errors it is it is

$$\tilde{w}^i \sim \exp\left(-0.5(\mathbf{y} - \mathbf{H}\mathbf{x}_i^f)^T \mathbf{R}^{-1} (\mathbf{y} - \mathbf{H}\mathbf{x}_i^f)\right)$$

The weights are also used for the transform matrix

$$\mathbf{W} = \sqrt{N}\left[\mathrm{diag}(\tilde{\mathbf{w}}) - \tilde{\mathbf{w}}\tilde{\mathbf{w}}^T\right]^{1/2} \Lambda$$

Here, Λ is the identity or a mean preserving random matrix that can be applied to stabilize the filter.

LETKF

In the LETKF we compute a local update of the ensemble mean and perturbations. The weight vector is computed according to the Kalman filter, which always assumes that the errors are Gaussian. Using the transform matrix

$$\mathbf{A}^{-1} = \rho(N-1)\mathbf{I} + (\mathbf{H}\mathbf{X}'^f)^T \mathbf{R}^{-1} \mathbf{H}\mathbf{X}'^f$$

that results from the equations of the Kalman filter and always assumes Gaussian errors we have

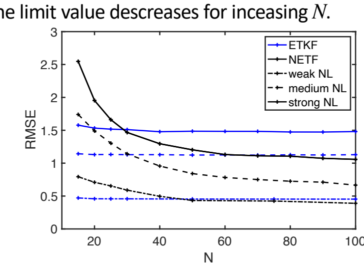
$$\tilde{\mathbf{w}} = \mathbf{A}(\mathbf{H}\mathbf{X}'^f)^T \mathbf{R}^{-1}\left(\mathbf{y} - \mathbf{H}\overline{\mathbf{x}^f}\right)$$
$$\mathbf{W} = \sqrt{N-1}\ \mathbf{A}^{1/2}\mathbf{\Lambda}$$

Experiments

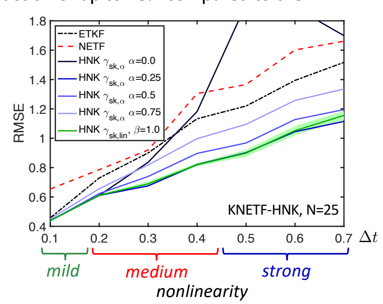
Lorenz-63 model

For the Lorenz-63 model, the default parameters are used. All 3 state variables are observed, The nonlinearity increases with the length of the forecast phase.

Below: ETKF and NETF for 3 different nonlinearities (NL). The NETF yields smaller errors for increasing N. The limit value descreases for inceasing N.



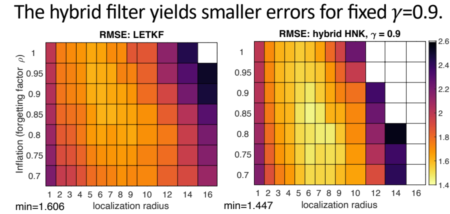
Below: RMS error for ETKF, NETF, and the hybrid filter variant HNK for different choices of γ and N=25. $\gamma_{sk,lin}$ leads to the smallest errors with an error reduction of up to 28% compared to the ETKF.



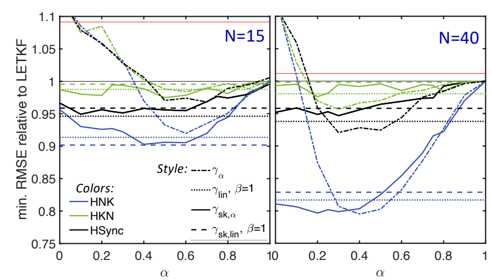
Lorenz-96 model

The results for the Lorenz-96 model (40 grid points, F=8) are shown for a forecast duration of 8 time steps. Each second grid point is observed.

Below: RMSe in dependence on localization radius and ensemble inflation for LETKF and LKNETF-HNK. The hybrid filter yields smaller errors for fixed γ=0.9.



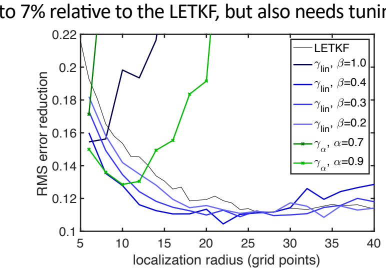
Below: $\gamma_{sk,lin}$ yields optimal (N=15) or nearly optimal (N=40) errors. The overall smallest errors are obtained with $\gamma_{sk,\alpha}$ for optimal tuning for N=40. the variant HNK yields smallest errors. Hybridization with skewness/kurtosis always reduces the errors compared to LETKF. Using only N_{eff} can increase the error for small α (γ_α).



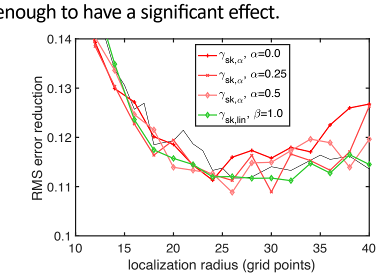
NEMO

NEMO is used in a double-gyre configuration (SEABASS in NEMO 3.3) with a resolution of 0.25°. Assimilated is simulated along-track SSH data with an observation error of 5 cm over two years with N=120. Observations are available each 2nd day.

Below: RMSE reduction of SSH by the assimilation when using N_{eff}**.** γ_α leads to filter divergence unless α is close to 1. γ_{lin} leads to an error reduction by up to 7% relative to the LETKF, but also needs tuning.



Below: RMSE reduction by the assimilation when using skewness and kurtosis: The skewness and kurtosis for the observed ensemble is not large enough to have a significant effect.



Hybrid Filter LKNETF

Due to the similarity of NETF and ETKF one can easily combine both filters into a hybrid analysis step. Different hybrid schemes can be formulated:

1-step update (*HSync*)

$$\mathbf{X}^a_{HSync} = \overline{\mathbf{X}}^f + (1-\gamma)\Delta\mathbf{X}_{NETF} + \gamma\Delta\mathbf{X}_{ETKF}$$

Here the analysis increments $\Delta\mathbf{X}$ of both filters are computed and then a weighted average of both is used.

2-step updates (HNK and HKN)

In the 2-step update we can compute the NETF first followed by the ETKF, both with increased observation errors according to the hybrid weight (Variant *HNK*):

Step 1: $\tilde{\mathbf{X}}^a_{HNK} = \mathbf{X}^a_{NETF}[\mathbf{X}^f, (1-\gamma)\mathbf{R}^{-1}]$

Step 2: $\mathbf{X}^a_{HNK} = \mathbf{X}^a_{ETKF}[\tilde{\mathbf{X}}^a_{HNK}, \gamma\mathbf{R}^{-1}]$

Alternatively, we can compute the ETKF update before the NETF (Variant *HKN*).

Summary

The hybrid ensemble filter LKNETF combines the stable LETKF with the second-order exact particle filter NETF. Different variants of the hybrid filter are introduced.

The assimilation experiments for all three models are implemented using PDAF [4,5] so that identical filter implementations are used. The hybrid variant HNK that applies the NETF to produce an intermediate result that is further used in the LETKF yields the lowest estimation errors. The hybrid rule basing on skewness and kurtosis yields very stable results and the lowest errors for the chaotic Lorenz models.

For NEMO at a resolution of 0.25°, the rules using the effective sample size yield the smallest errors. Here, the ensemble is not non-Gaussian enough to used the skewness and kurtosis to define the hybrid wiegth

References:

[1] Tödter, J. and B. Ahrens. *Mon. Wea. Rev.* (2015) **143**: 1347-1367

[2] Kirchgessner, P., J. Tödter, B. Ahrens, L. Nerger. *Tellus A* (2017) **69**: 1327766

[3] Frei, M. and Künsch, H. R. *Biometrika* (2013) **100**: 781–800

[4] Nerger, L. and Hiller, W. *Computers & Geosciences* (2013) **55**: 110–118

[5] http://pdaf.awi.de