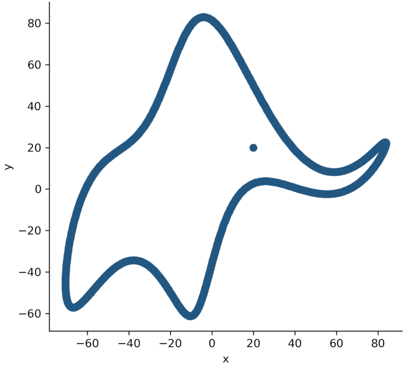
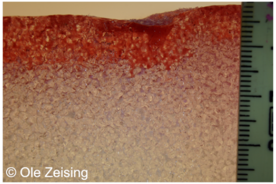
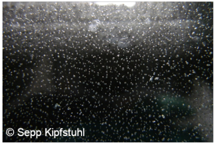
Towards a physics informed firn densification model

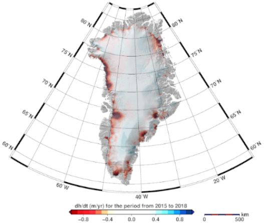


Mayer, J., Khairy, K., and Howard, J. (2010). *Drawing an elephant with four complex parameters.* American Journal of Physics, 78 (648), https://doi.org/10.1119/1.3254017.

Why modelling firn densification?

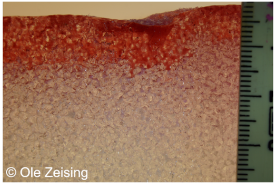


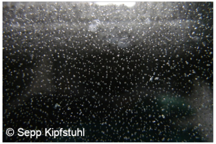


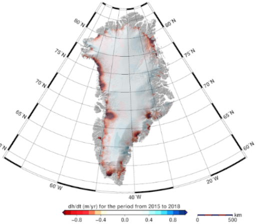


Why modelling firn densification?

- translation of height changes of the ice sheets to mass changes
- simulation of the pore close off, Δ-age
- understanding and simulating firn hydrology
- improving our understanding of the involved processes







The Model of Herron & Langway (1980)

$$\frac{\partial \rho}{\partial t} = c_0 \left(\rho_{\text{ice}} - \rho\right) \qquad \rho \leq 550\,\mathrm{kg\,m^{-3}}$$
$$\frac{\partial \rho}{\partial t} = c_1 \left(\rho_{\text{ice}} - \rho\right) \qquad \rho > 550\,\mathrm{kg\,m^{-3}}$$

Herron and Langway (1980). *Firn Densification: An Empirical Model.* Journal of Glaciology, 25(93), 373 – 385, https://doi.org/10.3189/S0022143000015239.

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$$c_0\left(\dot{b}, T\right) = \dot{b}^{\alpha}\ A_0 \exp\left(-\frac{Q_0}{R\,T}\right)$$
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$$c_0\left(\dot{b}, T\right) = \dot{b}^{\alpha}\ A_0 \exp\left(-\frac{Q_0}{R\,T}\right) \qquad \alpha = 1.0$$

$$c_1\left(\dot{b}, T\right) = \dot{b}^{\beta}\ A_1 \exp\left(-\frac{Q_1}{R\,T}\right) \qquad \beta = 0.5$$

TABLE II. VALUES OF THE CONSTANTS a AND b DERIVED FROM ACCUMULATION RATES AND SLOPES DERIVED FROM GRAPHS OF DEPTH AGAINST $\ln[\rho/(\rho_1-\rho)]$ FOR PAIRS OF SITES

Site pair	*a*	*b*
Site 2–RID	0.8	0.3
Site 2–Milcent	1.2	ID
Site 2–LAS	1.0	0.6
Site 2–C-7-3	1.1	0.6
C-7-3–LAS	1.2	0.7
C-7-3–Old Byrd	1.4	0.4
Old Byrd–J-9	1.4	0.3
RID–Milcent	0.9	0.5
RID–South Dome	ID*	0.3
LAS–Milcent	1.1	ID
Wilkes S2–Dye 3	ID	0.5
Crête–North Central	ID	0.6
Average and standard deviation	1.1±0.2	0.5±0.2

* ID = Insufficient data.

Herron and Langway (1980). *Firn Densification: An Empirical Model.* Journal of Glaciology, 25(93), 373 – 385, https://doi.org/10.3189/S0022143000015239.

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$$c_0\left(\dot{b}, T\right) = \dot{b}^{\alpha}\; A_0 \exp\left(-\frac{Q_0}{R\,T}\right) \qquad A_0 = 55, Q_0 = 10\,160\,\mathrm{J\,mol^{-1}}$$
$$c_1\left(\dot{b}, T\right) = \dot{b}^{\beta}\; A_1 \exp\left(-\frac{Q_1}{R\,T}\right) \qquad A_1 = 575, Q_1 = 21\,400\,\mathrm{J\,mol^{-1}}$$

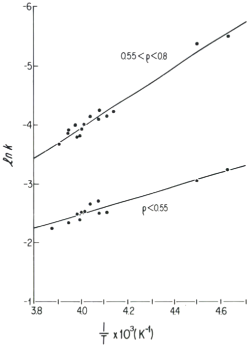


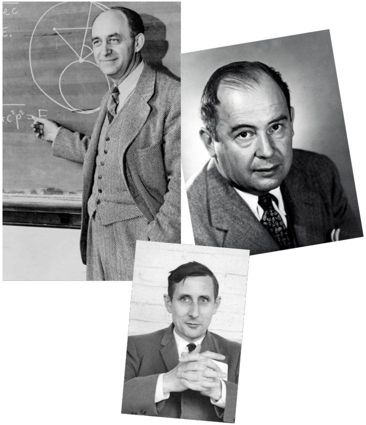
Fig. 1. Arrhenius plots for k_0 (lower line) and k_1 (upper line).

Herron and Langway (1980). *Firn Densification: An Empirical Model.* Journal of Glaciology, 25(93), 373 – 385, https://doi.org/10.3189/S0022143000015239.

Parameters

„I remember my friend Johnny von Neumann used to say, with four parameters I can fit an elephant, and with five I can make him wiggle his trunk.“

— Enrico Fermi to Freeman Dyson about John von Neumann



Freeman Dyson (2004). *A meeting with Enrico Fermi.* Nature, 427(297), https://doi.org/10.1038/427297a.
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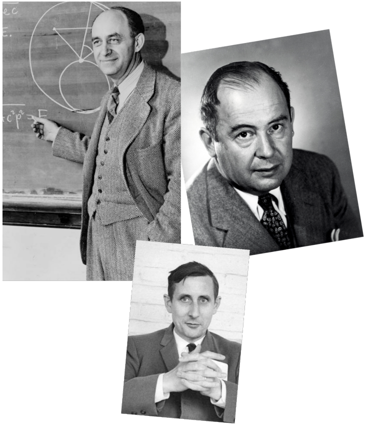

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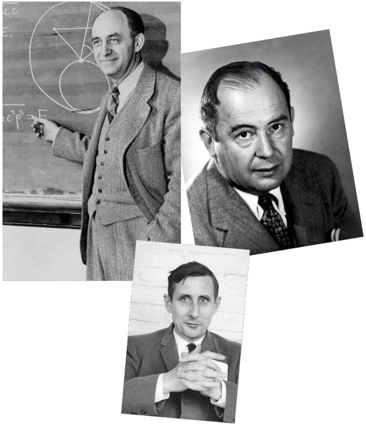

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Flavours of Herron & Langway (1980)

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Arthern et al. (2010)

$$c_0 = 0.07\,\dot{b}\,g\,\exp\left(-\frac{E_c}{R\,T} + \frac{E_g}{R\,T_{av}}\right)$$
$$c_1 = 0.03\,\dot{b}\,g\,\exp\left(-\frac{E_c}{R\,T} + \frac{E_g}{R\,T_{av}}\right)$$

Ligtenberg et al. (2011)

$$c_0 = \left(1.453 - 0.151\,\ln(\dot{b})\right)\,0.07\,\dot{b}\,g\,\exp\left(-\frac{E_c}{R\,T} - \frac{E_g}{R\,T_{av}}\right)$$
$$c_1 = \left(2.366 - 0.293\,\ln(\dot{b})\right)\,0.03\,\dot{b}\,g\,\exp\left(-\frac{E_c}{R\,T} - \frac{E_g}{R\,T_{av}}\right)$$

Simonsen et al. (2013)

$$c_0 = f_0\;0.07\,\dot{b}\,g\,\exp\left(-\frac{E_c}{R\,T} - \frac{E_g}{R\,T_{av}}\right)$$
$$c_1 = f_1\;0.03\,\dot{b}\,g\,\exp\left(-\frac{E_c}{R\,T} - \frac{E_g}{R\,T_{av}}\right)$$

Kuipers Munneke et al. (2015)

$$c_0 = \left(1.042 - 0.0916\,\ln(\dot{b})\right)\,0.07\,\dot{b}\,g\,\exp\left(-\frac{E_c}{R\,T} - \frac{E_g}{R\,T_{av}}\right)$$
$$c_1 = \left(1.734 - 0.2039\,\ln(\dot{b})\right)\,0.03\,\dot{b}\,g\,\exp\left(-\frac{E_c}{R\,T} - \frac{E_g}{R\,T_{av}}\right)$$

Medley et al. (2020), in review

$$c_0 = 0.07\,\dot{b}^{\alpha}\,g\,\exp\left(-\frac{E_{c_0}}{R\,T} + \frac{E_g}{R\,T}\right)$$
$$c_1 = 0.03\,\dot{b}^{\beta}\,g\,\exp\left(-\frac{E_{c_1}}{R\,T} + \frac{E_g}{R\,T}\right)$$

Zwally & Li (2002)

$$c = \dot{b}^{\alpha}\;\beta\,K_{0G}\,\exp\left(-\frac{E}{R\,T}\right)$$

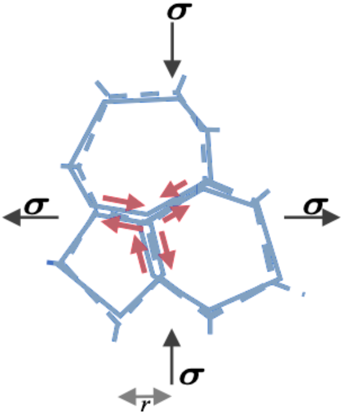
Li & Zwally (2011)

$$c_0 = -9.788 + 8.996\,\bar{b} - 0.6165\,T_m\,8.36\,(273.2 - T)^{-2.061}\,\bar{\dot{b}}$$
$$c_1 = c_0 / \left(-2.0178 + 8.4043\,\bar{b} - 0.0932\,T_m\right)$$

Li & Zwally (2015)

$$c_0 = -1.218 - 0.403\,T_m\,8.36\,(273.2 - T)^{-2.061}\,\bar{\dot{b}}$$
$$c_1 = c_0 \cdot \left(0.792 - 1.080\,\bar{b} + 0.00465\,T_m\right)$$

Grain Boundary Sliding



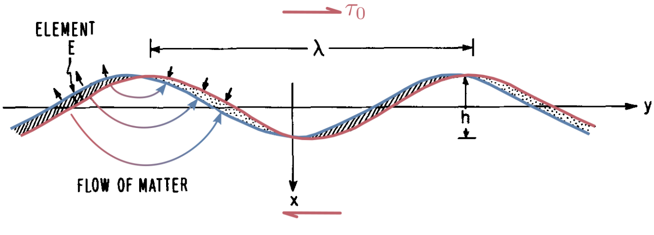


Fig. 3.1—Steady-state sliding with diffusional accommodation.

Raj, R. and Ashby, M. F. (1971). *On Grain Boundary Sliding and Diffusional Creep.* Metallurgical Transactions, 2, 1113–1127.

Grain Boundary Sliding

$$\dot{\varepsilon}_{zz} = -\frac{2}{15} \quad \delta_b \quad \frac{8\, D_{\mathrm{BD}}\, \Omega}{k_b\, T\, h^2} \quad \frac{1}{r\, \mu^2} \quad \left(\frac{\rho_{\mathrm{ice}}}{\rho}\right)^3 \quad \left(1 - \frac{5}{3}\frac{\rho}{\rho_{\mathrm{ice}}}\right) \quad \sigma_{zz}$$

$$D_{\mathrm{BD}} = A_{\mathrm{BD}} \exp\left(-\frac{Q_{\mathrm{BD}}}{R\, T}\right)$$

$\dot{\varepsilon}_{zz}$	vertical strain rate	k_b	Boltzmann's constant	μ	ratio grain radius / neck radius
δ_b	width of grain boundary	T	Temperature	ρ_{ice}	ice density
D_{BD}	boundary sliding	h	amplitude grain boundary obstructions	ρ	firn density
Ω	volume of H_2O molecule	r	grain radius	σ_{zz}	stress in vertical direction
A_{BD}	boundary diffusion coefficient	Q_{BD}	activation energy boundary diffusion	R	gas constant

Alley, R. B. (1987). *Firn Densification by Grain-Boundary Sliding: A First Model.* Journal de Physique Colloques, 48(C1), C1-249–C1-256.

Variants

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Variant 1

$$\dot{\varepsilon}_{zz\,\mathrm{v}_1} = -C_{\mathrm{v}_1}\, D_{\mathrm{BD}}\, \frac{1}{T}\, \frac{1}{r} \left(\frac{\rho_{\mathrm{ice}}}{\rho}\right)^3 \left(1 - \frac{5}{3}\, \frac{\rho}{\rho_{\mathrm{ice}}}\right) \sigma_{zz}$$

Variant 2

$$\dot{\varepsilon}_{zz\,\mathrm{v}_2} = -C_{\mathrm{v}_2}\, D_{\mathrm{BD}}\, \frac{1}{T}\, \frac{1}{r} \left(\frac{\rho_{\mathrm{ice}}}{\rho}\right)^3 \left(1 + \frac{0.5}{6} - \frac{5}{3}\, \frac{\rho}{\rho_{\mathrm{ice}}}\right) \sigma_{zz}$$

Variant 3

$$\dot{\varepsilon}_{zz\,\mathrm{v}_3} = -C_{\mathrm{v}_3}\, \frac{1}{T}\, \frac{1}{r} \left(\frac{\rho_{\mathrm{ice}}}{\rho}\right)^3 \left(1 - \frac{5}{3}\, \frac{\rho}{\rho_{\mathrm{ice}}}\right) \sigma_{zz}$$

Variant 4

$$\dot{\varepsilon}_{zz\,\mathrm{v}_4} = -C_{\mathrm{v}_4}\, \frac{1}{T}\, \frac{1}{r} \left(\frac{\rho_{\mathrm{ice}}}{\rho}\right)^3 \left(1 + \frac{0.5}{6} - \frac{5}{3}\, \frac{\rho}{\rho_{\mathrm{ice}}}\right) \sigma_{zz}$$

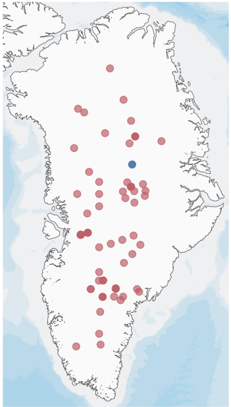
Locations

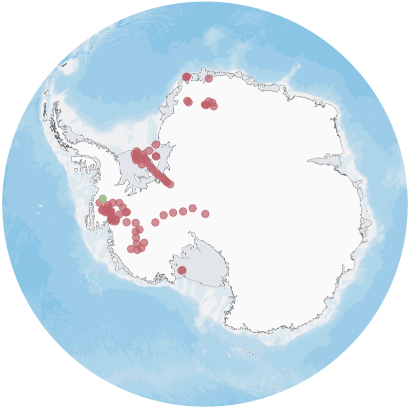
159 firn profiles

Greenland: 80
Antarctica: 79

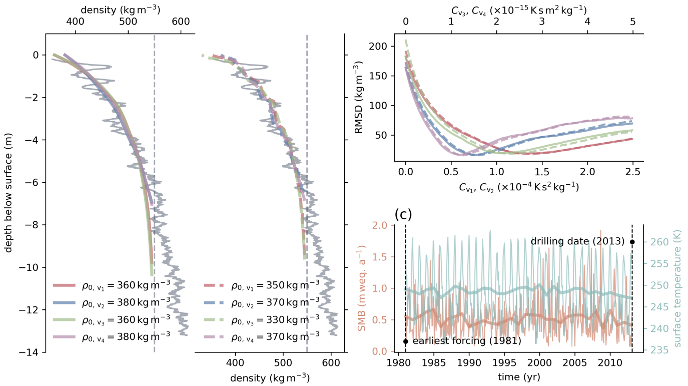
firn profiles
Koenig, L. and Montgomery, L. (2019). *Surface Mass Balance and Snow Depth on Sea Ice Working Group (SUMup) snow density subdataset, Greenland and Antarcitca, 1950–2080*. Arctic Data Center.

map data
Amante, C. and Eakins, B. W. (2009). *ETOPO1 1 Arc-Minute Global Relief Model: Procedures, Data Source, and Analysis*. NOAA Technical Memorandum NESDIS NGDC-24, National Geophysical Data Center, NOAA.
Arndt, J. E., Schenke, H. W., Jakobsson, M., ...(2013). *The International Bathymetric Chart of the Southern Ocean (IBSCO) Version 1.0 – A new bathymetric compilation covering circum-Antarctic waters*. Geophys. Res. Lett., 40, 3111–3117.





iSTAR site 3



firn density

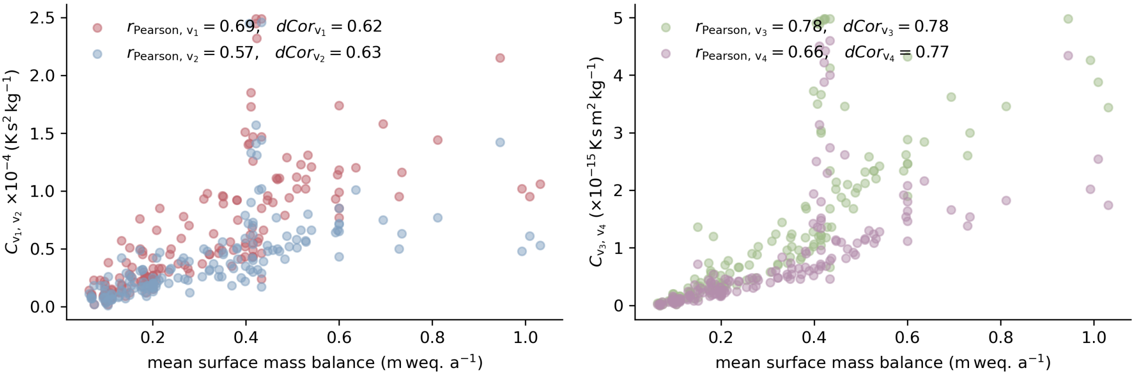
Morris, E. M., Mulvaney, R., Arthern, R. J., …(2017). *Snow Densification and Recent Accumulation Along the iSTAR Traverse, Pine Island Glacier, Antarctica*. J. Geophys. Reas.-Earth, 122, 2284–2301.

forcing

Muñoz Sabater, J. (2019). *ERA5-Land hourly data from 1981 to Present*. Copernicus Climate Change Service (C3S) Climate Data Store (CFS).

Hersbach, H., Bell, B, Berrisford, P., …(2020). *The ERA5 global reanalysis*. Q. J. Roy. Meterol. Soc., 146, 1999–2049.

Correlation with SMB



Compressibility

$$\dot{\varepsilon}_{zz} = -\frac{2}{15} \quad \delta_b \quad \frac{8\, D_{\mathrm{BD}}\, \Omega}{k_b\, T\, h^2} \quad \frac{1}{r\, \mu^2} \quad \left(\frac{\rho_{\mathrm{ice}}}{\rho}\right)^3 \quad \left(1 - \frac{5}{3}\, \frac{\rho}{\rho_{\mathrm{ice}}}\right) \quad \sigma_{zz}$$

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$$\boldsymbol{T} = -p\,(\boldsymbol{x}, t)\,\mathbf{1} + 2\eta \boldsymbol{D}$$

$\boldsymbol{T}$	Cauchy stress tensor	p	thermodynamic pressure
ρ	mass density	η	shear viscosity
$\boldsymbol{D}$	spatial strain rate tensor	$\mathbf{1}$	identity tensor
λ	Lamé's first parameter	tr	trace

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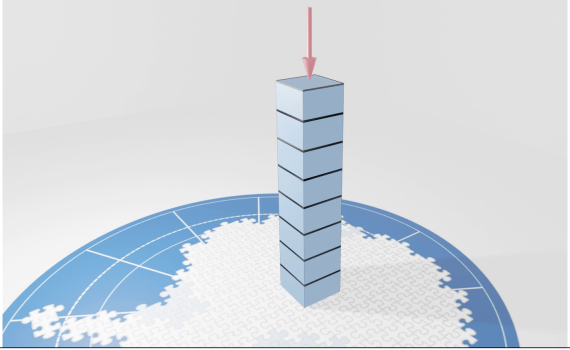

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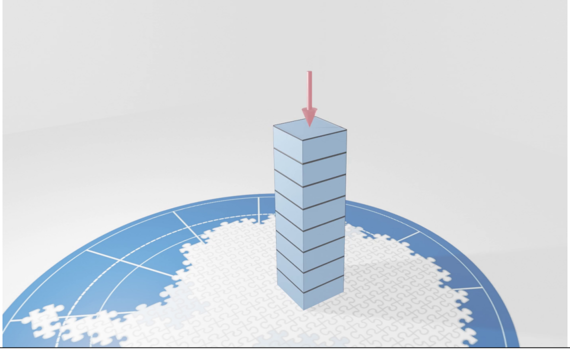

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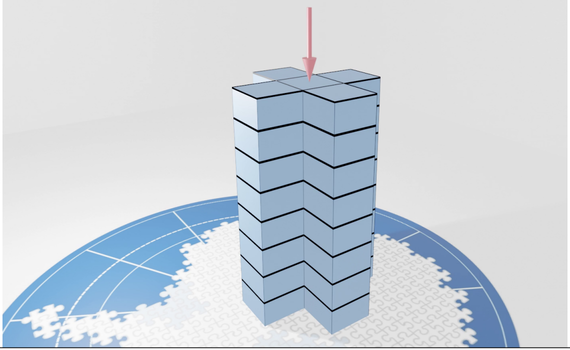

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$$\boldsymbol{T} = -p\,(\boldsymbol{x}, t)\,\mathbf{1} + 2\eta \boldsymbol{D} + \lambda\,(\mathrm{tr}\,\boldsymbol{D})\,\mathbf{1}$$

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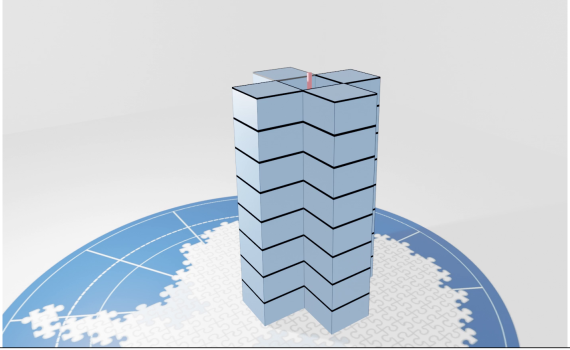

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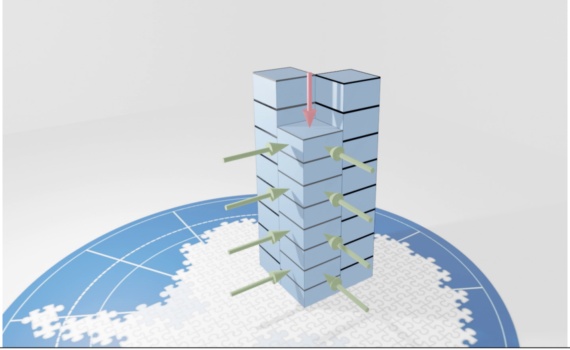

Thank You

Schultz, T., Müller, R., Gross, D., and Humbert, A. (2022). *On the contribution of grain boundary sliding type creep to firn densification – an assessment using an optimization approach.* The Cryosphere, 16, 143–158, https://doi.org/10.5194/tc-16-143-2022.

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